Principal Series of GL(2) Over a Finite Field

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Abstract

This thesis identifies irreducible representations of $\mathrm{GL}(2,k)$ for a finite field k. For certain subgroups M, N, and P of $\mathrm{GL}(2,k)$, M-representations are extended trivially to P-representations and then induced to give rise to principal series representations. We demonstrate that the representations found by examining principal series account for roughly half of the irreducibles of $\mathrm{GL}(2,k)$. Unlike in classical presentations, the algebraic techniques employed here have roles in the study of infinite-dimensional representations of matrix groups over infinite fields.

Chapter 1

Introduction

All representations are assumed to be linear and finite-dimensional. Given the category we are in, all maps will be homomorphisms whenever it is sensible for them to be so (and note that homomorphisms between vector spaces are *linear* maps). In general, G denotes a group, V denotes a G-representation, and k denotes a finite field of order q. Ideally, all of the material here should be accessible to an advanced undergraduate with a background in group theory.

The goal of this thesis is to identify irreducible representations of $\mathrm{GL}(2, k)$. Nearly all of the methods employed apply to $\mathrm{SL}(2, k)$ as well, as shall be noted throughout the document. However, the irreducible representations found in the end (Section 3.3) will specifically be representations of the general linear group.

Three significant subgroups of $\mathrm{GL}(2, k)$ are the *standard Levi component* M, the *unipotent radical* N, and the *parabolic group* P, where $P = MN$ and M normalizes N. We trivially extend M-representations to P-representations and then induce to yield *principal series* representations of $\mathrm{GL}(2, k)$. Chapter 3 showcases that the representations found by examining principal series will account for roughly half of the irreducible representations of $\mathrm{GL}(2, k)$. Roughly the other half of irreducibles can be found by extending N-representations to P-representations (in a less trivial way) and then inducing; however, we will only focus on finding the irreducibles extended from M.

Pedagogically this thesis motivates why we investigated principal series to find irreducible $\mathrm{GL}(2, k)$ representations. Chapter 2 demonstrates how intertwining operators between principal series expose irreducible representations, which causes such intertwining operators to be examined directly in Chapter 3. The hope is that by placing insightful results in a natural order, this thesis communicates *intuition* and *intent*.

The irreducible representations of $\mathrm{GL}(2, k)$ are well understood [2, 4, 7, 8, 14]. However, unlike classical presentations of the material, this thesis examines irreducible representations using methods that *scale up*. That is, the algebraic techniques play a part in the study of infinite-dimensional representations of matrix groups over infinite fields, where analysis and topology are also required. Although we do not engage with the analysis and topology, and inevitably we invoke finite dimensionality, the goal here is to use forward-looking methods.

For the careful reader, Appendix C contains small, technical proofs that were used throughout this thesis (denoted [C.1] through [C.6]). These proofs are not essential for the narrative; however, they are provided for completeness.

1.1 Relevant Definitions

Definition 1.1. Given a group G and a set S, define an *action of G on S* to be a map

$$G \times S \longrightarrow S, \quad (g, s) \longmapsto g \cdot s,$$

where $\cdot$ is any operation such that

$$\begin{aligned} 1_G \cdot s &= s && \text{for all } s \in S && \text{(identity-compatibility)} \\ (gg') \cdot s &= g \cdot (g' \cdot s) && \text{for all } g, g' \in G, s \in S && \text{(associativity).} \end{aligned}$$

Definition 1.2. A *linear representation* of a group G is a complex vector space V and a group homomorphism

$$\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V), \qquad g \longmapsto \left(v \overset{\rho(g)}{\longmapsto} g \cdot v\right)$$

such that the G-action on V is linear. We say the representation of G is *over* V. The *dimension* of a representation refers to $\dim_{\mathbb{C}}(V)$. In practice, we specify a representation completely by stating its representation space V and a G-action; the map above is then understood.

The linearity of the G-action ensures that $\rho(g) \in \mathrm{Aut}_{\mathbb{C}}(V)$. Note that the group operation of $\mathrm{Aut}_{\mathbb{C}}(V)$ is composition. A common abuse of notation is to say "$v \in \rho$" rather than "$v \in V$" to emphasize the action by ρ over the representation space V.

Furthermore, a linear G-action on a complex vector space V corresponds exactly to a linear $\mathbb{C}[G]$-action on V and vice-versa.[1] For this reason, V can also be thought of as a $\mathbb{C}[G]$-module, yielding an equivalent definition.

Definition 1.3. (Equivalent to Definition 1.2). A *linear representation* of a group G is a complex vector space V that is also a $\mathbb{C}[G]$-module.

However, for consistency, we will usually invoke Definition 1.2 to describe representations. Naturally, we might imagine how a representation (V, ρ) might have subspaces of V that are also representations.

Definition 1.4. A *subrepresentation* of a G-representation (V, ρ) is a G-invariant subspace $W \subseteq V$. That is,

$$g \cdot w \in W \text{ for all } w \in W.$$

Note that the G-action on W is simply a restriction of the G-action on V.

[1]Specifically, let $\{e_g\}_{g \in G}$ be a basis of indicator functions of $\mathbb{C}[G]$; that is, $e_g(x)$ is 1 if $x = g$ and 0 otherwise. Define "$e_g \cdot v$" to be $(g \cdot v)$ for $v \in V$. Extend the basis action linearly to get a $\mathbb{C}[G]$-action.

Furthermore, if W is a subrepresentation of V, the G-action on V *descends* to an action on the quotient space V/W.

Definition 1.5. Given a subrepresentation (W, ρ) of (V, ρ), define the *quotient representation* to have space V/W and G-action

$$g \cdot (v + W) = g \cdot v + W \quad (g \in G, v \in V).$$

We confirm that the proposed action is well-defined. If $v + W = v' + W$, then $v - v' \in W$, so by G-invariance, $g \cdot (v - v') \in W$. By linearity, $g \cdot v + W = g \cdot v' + W$, as desired. The proposed operation is indeed a G-action, since associativity and identity-compatibility immediately follow from V having a defined G-action.

Recall that it is often possible to decompose a vector space into a direct sum of eigenspaces. Similarly, we can sometimes decompose a G-representation into a direct sum of subrepresentations.

Definition 1.6. A representation (V, ρ) is *irreducible* if it has no proper, nontrivial subrepresentations. A representation whose space can be decomposed into a direct sum of irreducible representations is said to be *completely reducible*.

Section 1.3 discusses complete reducibility in more detail. We now consider what it means for representations to be isomorphic.

Definition 1.7. Two representations (V_1, ρ_1) and (V_2, ρ_2) are said to be *isomorphic* if there exists an isomorphism $\Phi : V_1 \longrightarrow V_2$ that intertwines the action of G; that is,

$$\Phi(\rho_1(g)(v)) = \rho_2(g)(\Phi(v)).$$

Equivalently, we see that Φ is a linear G-map[2] from V_1 to V_2.

Given Definition 1.7, we can now formulate the basic problem of representation theory:

Classify all representations of a given group G up to isomorphism.

This thesis will undergo that task for $G = \mathrm{GL}(2, k)$ where k is a finite field of order q. As previously noted, nearly all of the methods employed can be used to find representations of $\mathrm{SL}(2, k)$ as well. However, the representations actually found in the end will be of the general linear group. We will discover several types of nonisomorphic G-representations by studying *principal series* representations of G, which will be introduced in Section 1.7.

1.2 Examples of Representations

Here, we illustrate examples of representations so that the former definitions may become live. The first two examples are fairly straight-forward, and the last involves some review of linear algebra.

[2]By a *G-map*, we mean that $\Phi(g \cdot v) = g \cdot \Phi(v)$ for all $g \in G$. Also, Φ is linear since it is an isomorphism between vector spaces.

01° *Characters are one-dimensional representations.*

Definition 1.8. A *character* χ is a homomorphism from a (finite) group to the multiplicative group of $\mathbb{C}$.

The literature on representation theory [2, 7, 8, 14] often defines a character of a representation (V, ρ) to be the trace of the matrix $\rho(g)$. *Here, a character does not refer to this specific map*; instead, it can refer to any homomorphism from a finite G to $\mathbb{C}^\times$.

Give a character $\chi : G \longrightarrow \mathbb{C}^\times$, we can naturally define a G-representation with a one-dimensional representation space ($\mathbb{C}$) and G-action $g \cdot v = \chi(g)v$.

02° $\mathbb{C}[G]$ *is a representation of* G.

Let G be a finite group. On one hand, we mentioned before that a G-representation also has a $\mathbb{C}[G]$-action. In addition, we can define $\mathbb{C}[G]$ to have a G-action and hence be a G-representation. Let $\{e_g\}_{g\in G}$ be a basis of indicator functions of $\mathbb{C}[G]$; that is, $e_g(x) = 1$ if $x = g$ and 0 otherwise. Define the G-action on a basis element to be

$$g \cdot e_h = e_{gh} \quad (\text{for } g, h \in G)$$

and extend this action linearly to $\mathbb{C}[G]$.

03° *Representations of a cyclic group decompose into characters.*

Let G be the cyclic group of n elements with generator g. Let V be a finite-dimensional, complex vector space. To find the representations of G, we need a group homomorphism

$$\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V).$$

The image of G under ρ is completely determined by $\rho(g)$ for generator g of G. By choosing a basis β of V, we can write the matrix $\rho(g)$ with respect to that basis. Selecting β appropriately will shed insight as to what the representations of G are.

The structure theorem for modules over a PID [13] shows [9] that there exists a *Jordan basis* such that $\rho(g)$ with respect to that basis has block-diagonal *Jordan normal form.* We elaborate in the next two definitions.

Definition 1.9. A *Jordan block* with value $\lambda \in \mathbb{C}$ is a square, upper-triangular matrix whose entries are λ on the diagonal, 1 immediately above the diagonal, and 0 everywhere else.

For instance, the following matrices are Jordan blocks,

$$\begin{bmatrix}\lambda\end{bmatrix}, \begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}, \begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{bmatrix}.$$

Definition 1.10. A *Jordan form* matrix is a block diagonal matrix whose blocks are all Jordan blocks.

As an example, following is a Jordan form matrix with its Jordan blocks distinguished for clarity,

$$\left[\begin{array}{cc|cc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ \hline 0 & 0 & 0 & 0 & 5 \end{array}\right].$$

By selecting β appropriately, we have Jordan blocks $J_1, \ldots, J_m$ such that $[\rho(g)]_\beta$ is

$$\begin{bmatrix} J_1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & J_m \end{bmatrix}.$$

Note that $1 \leq m \leq \dim(V)$. Since G is cyclic and ρ is a homomorphism,

$$\rho(g)^n = \overbrace{\rho(g) \circ \cdots \circ \rho(g)}^{n} = \rho(g^n) = \rho(1_G) = I_V.$$

Also, $\big([\rho(g)]_\beta\big)^n$ will be a block diagonal matrix with block diagonals J_k^n. Thus, each block J_k^n must be an identity matrix.

For each J_k, let N be the (nilpotent) Jordan block with $\lambda = 0$ such that $\dim(J_k) = \dim(N)$. That is, N is a matrix with 1's above its diagonal and 0's everywhere else. Then $J_k = \lambda I + N$, where I is the appropriately-sized identity matrix.

By expanding $(J_k)^n = (\lambda I + N)^n$ with the binomial theorem, one can easily see that J_k^n is the identity if and only if N is the 1×1 zero-matrix (see [14] for a formal proof). Since $\dim(J_k) = \dim(N)$, we conclude that $J_k = [\lambda]$. Furthermore, since $J_k^n = I$, we have that $\lambda^n = 1$. Thus, $[\rho(g)]_\beta$ is a *diagonal matrix* whose diagonal entries are any m of the nth roots of unity,

$$[\rho(g)]_\beta = \begin{bmatrix} \zeta_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \zeta_m \end{bmatrix}.$$

So $[\rho(g^\ell)]_\beta$ for some $g^\ell \in G$ will be diagonal with diagonal entries $\zeta_1^\ell, \ldots, \zeta_m^\ell$. Then on each one-dimensional subspace of V spanned by an element of β, the G-action is

$$g^\ell \cdot [v]_\beta = [\rho(g^\ell)]_\beta [v]_\beta = \zeta_k^\ell [v]_\beta$$

for some root of unity ζ_k. Since any one-dimensional subspace of V is a copy of $\mathbb{C}$, we have that $\mathbb{C}$ is a representation of G with

$$\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}), \quad \rho(g^\ell)(z) = \zeta_k^\ell\, z.$$

So, the character $\chi : G \longrightarrow \mathbb{C}^\times$ such that $\chi(g^\ell) = \zeta_k^\ell$ for some ζ_k describes the G-action, and a representation of V decomposes into such characters. The next section explores the possibility of decomposing other representations into characters.

1.3 Complete Reducibility

The goal of this section is to prove the next proposition. In addition, we prove several results (Lemma 1.16, Corollary 1.17, Corollary 1.19) that are frequently invoked throughout the thesis.

Proposition 1.11. (Complete Reducibility). Every representation of a finite group G over $\mathbb{C}$ is completely reducible.

In order to prove Complete Reducibilty, we will first introduce the notion of unitary representations.

Definition 1.12. A representation (V, ρ) is *unitary* if V has a hermetian inner product $\langle\ ,\ \rangle$ which preserves G-action; symbolically,

$$\langle v, w \rangle = \langle g \cdot v, g \cdot w \rangle \quad \text{for all } v, w \in V, g \in G.$$

A representation is *unitarisable* if it can be equipped with such an inner product.

Lemma 1.13. Let (V, ρ) be a unitarisable representation of G, and let W be a G-invariant subspace of V. Then $W^\perp$ is also a G-invariant subspace of V.

Proof. Since $W^\perp$ is a subspace of V, we just need to check that $W^\perp$ is G-invariant. Let $x \in W^\perp$. For $g \in G$, since W is G-invariant, $g^{-1} \cdot w \in W$ for all $w \in W$. Then,

$$\begin{aligned} 0 &= \langle g^{-1} \cdot w, x \rangle && [\text{def. of } W^\perp] \\ &= \langle g \cdot (g^{-1} w), g \cdot x \rangle && [V \text{ is unitary}] \\ &= \langle w, g \cdot x \rangle && [\text{action is associative.}] \end{aligned}$$

Since $w \in W$ was arbitrary, we see that $g \cdot x \in W^\perp$, so $W^\perp$ is G-invariant. □

Thus, for unitarisable (V, ρ) with subrepresentation W, we have that subspace $W^\perp \subseteq V$ is *also* a subrepresentation. From linear algebra, if V is finite-dimensional, we can write

$$V = W \oplus W^\perp.$$

However, we now see that this is a decomposition of *representations* rather than just a decomposition of subspaces. As a result, we have the following theorem.

Theorem 1.14. *A finite-dimensional, unitary representation (V, ρ) admits a decomposition into irreducible, unitary subrepresentations.*

Proof. For any subrepresentation W of V, give W the same inner-product as V, only restrict its inputs to be elements of W. By G-invariance of W, the inner-product will be W-invariant; hence, W is unitarisable. Since W was arbitrary, we just need to show that V decomposes into irreducible subrepresentations.

If V is irreducible, then the statement is trivial. Thus, suppose V is reducible. Then V has a proper, nontrivial subrepresentation W. By Lemma 1.13, V decomposes as $V = W \oplus W^\perp$ for (unitary) subrepresentations $W, W^\perp$. Furthermore,

$\dim(W), \dim(W^\perp) < \dim(V)$, since $W, W^\perp$ are proper subspaces. If W or $W^\perp$ is reducible, since each is a unitary, finite-dimensional representation, we can apply Lemma 1.13 again as needed. Continuing in this manner, by finite-dimensionality, the process *must* terminate, yielding a decomposition of V into irreducible subrepresentations. □

Thus, Theorem 1.14 proves Complete Reducibility *given that our representation of G is unitarisable*. However, the next theorem (which is commonly called "Weyl's unitary trick") yields the desired result.

Theorem 1.15. *Finite-dimensional representations of finite groups are unitarisable.*

Proof. Let (V, ρ) be a finite-dimensional representation of a finite group G. Then, define

$$\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{C}, \qquad \langle v, w\rangle = \frac{1}{|G|}\int_G \langle g\cdot v, g\cdot w\rangle_V \; dg \quad (g \in G)$$

where $\langle\ ,\ \rangle_V$ is the standard inner product on V and dg indicates *Haar measure* (counting measure on finite groups). Note that since G is finite, the integral above is equivalent to a finite sum. We use integral notation to suggest how to *scale up*, in which case we would integrate over general topological groups.[3]

We check that $\langle\ ,\ \rangle$ is a G-invariant inner product. Positive-definiteness, symmetry, and bilinearity of $\langle\ ,\ \rangle$ follow immediately from those properties of $\langle\ ,\ \rangle_V$. Furthermore, observe that for all $\tilde{g} \in G$,

$$\begin{aligned}
\langle \tilde{g}\cdot v, \tilde{g}\cdot w\rangle &= \frac{1}{|G|}\int_G \langle g\cdot(\tilde{g}v), g\cdot(\tilde{g}w)\rangle_V \; dg \\
&= \frac{1}{|G|}\int_G \langle (g\tilde{g})\cdot v, (g\tilde{g})\cdot w\rangle_V \; dg && \text{[associativity of action]} \\
&= \frac{1}{|G|}\int_G \langle g'\cdot v, g'\cdot w\rangle_V \; dg' && \text{[letting } g' = g\tilde{g}] \\
&= \langle v, w\rangle.
\end{aligned}$$

In the third equality, $dg = dg'$ since Haar measure is translation-invariant. The last equality follows, because integrating over all elements of G is *the same as* multiplying each element of G by some $g \in G$ and integrating over the result; this is because multiplication by $g \in G$ simply permutes the group. Thus, G is unitarisable. □

Thus, Theorems 1.14 and 1.15 combine to prove Complete Reducibility. Finally, we present a central result of Issai Schur (1905) [2, 4, 14].

Lemma 1.16. (Schur's lemma). If (V, ρ) is a complex, irreducible G-representation, then $\mathrm{Aut}_G(V) = \mathbb{C}$; that is, the only G-automorphisms of V are multiplications by scalars.[4]

[3] For such topological groups, the functions being integrated would decay rapidly or have compact support.

[4] This is a slight abuse of notation. By "$\mathbb{C}$", we mean "multiplication by scalars in $\mathbb{C}$"; for instance, $5 \in \mathbb{C}$ would be identified with *the map that scales by 5.*

Proof. Let $T \in \mathrm{Aut}_G(V)$. Since V is a complex vector space, the characteristic polynomial of T splits; so T has some eigenvector v_0 with eigenvalue λ. Compute that

$$\begin{aligned} T(g \cdot v_0) &= g \cdot T(v_0) && \text{[since } T \text{ is a } G\text{-map]} \\ &= g \cdot \lambda v_0 && \text{[since } T(v_0) = \lambda v_0] \\ &= \lambda(g \cdot v_0) && \text{[since } \lambda \text{ is a scalar.]} \end{aligned}$$

Since T is linear, $T = \lambda I_V$ on the subrepresentation of V generated by v_0. However, since V is irreducible (and this subrepresentation is clearly nontrivial), $T = \lambda I_V$ on all of V. Thus, the only G-automorphisms of V are multiplications by scalars. □

Corollary 1.17. Let χ_1, χ_2 be characters of G. Then

$$\mathrm{Hom}_G(\chi_1, \chi_2) = \begin{cases} \mathbb{C} & \text{if } \chi_1 = \chi_2 \\ 0 & \text{else.} \end{cases}.$$

Note that "0" denotes the zero vector space of $\mathrm{End}_{\mathbb{C}}(\mathbb{C})$ and "$\mathbb{C}$" again denotes all multiplication-by-scalar maps. Furthermore, by a hom-group between characters, we mean a hom-group between the natural representation spaces of those characters ($\mathbb{C}$) such that the G-action on each space is defined by χ_1 and χ_2, respectively. (This is a common abuse of notation).

Proof. For $f \in \mathrm{Hom}(\chi_1, \chi_2)$ and $g \in G, x \in \mathbb{C}$,

$$\chi_1(g)f(x) = f(\chi_1(g)x) = \chi_2(g)f(x),$$

which follows from f being linear (with scalar $\chi_1(g)$) and f being a G-map. Since the equality holds for all $x \in \mathbb{C}$, either $\chi_1 = \chi_2$ or $f = 0$. If $f \neq 0$, f is invertible and $\chi_1 = \chi_2$, so $\mathrm{Hom}_G(\chi_1, \chi_2) = \mathrm{Aut}_G(\chi_1) = \mathbb{C}$ by Schur's lemma. □

With Schur's lemma, the following proposition can be proved.

Theorem 1.18. *An irreducible, complex representation of a finite group with abelian action is one-dimensional.*

Proof. Let G be a finite, and let (V, χ) be an irreducible, complex representation of G. Then, by definition of action, there is a group homomorphism

$$G \xrightarrow{\chi} \mathrm{Aut}_{\mathbb{C}}(V), \quad g \mapsto (v \overset{\chi(g)}{\longmapsto} g \cdot v).$$

We will show that $\chi(g)$ is a G-map[5] since the G-action is abelian. Observe that for all $g, g' \in G$ and $v \in V$,

$$\begin{aligned} g' \cdot \chi(g)(v) &= g' \cdot (g \cdot v) = (g'g) \cdot (v) && \text{[def. of } \chi(g)] \\ &= (gg') \cdot v = g \cdot (g' \cdot v) && [G\text{-action is abelian}] \\ &= \chi(g)(g' \cdot v). \end{aligned}$$

[5]We know that χ is a G-homomorphism and $\chi(g)$ is a homomorphism but not necessarily that $\chi(g)$ is a G-homomorphism.

Thus, $\chi(g)$ is a G-homomorphism so $\chi(g) \in \mathrm{Aut}_G(V)$. Since G is irreducible, by Schur's lemma, $\chi(g)$ is multiplication by some scalar. Thus, any proper subspace of V will invariant under G-action and thus a proper G-subrepresentation. Since V is irreducible, V is then forced to have no proper subspaces, meaning $\dim_{\mathbb{C}}(V) = 1$. □

Note that if G is abelian, the G-action on V is abelian by default. Thus, we have the following corollary.

Corollary 1.19. An irreducible, complex representation of a finite, abelian group is one-dimensional.

Note that decomposition into one-dimensional subrepresentations means that characters define the action on each subrepresentation (as we saw in the example of the cyclic group decomposition). Furthermore, if V has trivial G-action, we have the following corollary to Complete Reducibility.

Corollary 1.20. Let G be a finite abelian group. Let 1_G denote the trivial G-representation; that is, 1_G is a copy of $\mathbb{C}$ with trivial G-action.[6] If (V, ρ) is a complex G-representation with trivial G-action, then $V = \bigoplus 1_G$.

Proof. By Complete Reducibility, V decomposes as a (finite) direct sum of irreducible, one-dimensional subrepresentations; that is, each subrepresentation is a copy of $\mathbb{C}$. If any of these subrepresentations has nontrivial G-action, then so does V, so each must have trivial G-action. □

The next section introduces the general environment of the thesis along with characters of significant groups in that environment.

1.4 General Environment

Let k be a finite field of order q. Let $G = \mathrm{GL}(2, k)$ or $G = \mathrm{SL}(2, k)$. Define the following subgroups of G:

$$M = \left\{ \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \right\}, \quad N = \left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\}, \quad P = \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\},$$

where $*$ indicates any element of k such that the determinants of the matrices are appropriate given the group to which they belong. These subgroups of G are named: the *standard Levi component*, the *unipotent radical*, and the *standard parabolic subgroup*, respectively. It is easily verified that M and N are abelian, so by Complete Reducibility, representations of either decompose into characters. Therefore, it is natural to see how the characters of M and N yield information about the representations of P and G.

Note that N is normal in P and M normalizes N (so that $mnm^{-1} \in N$ for all $m \in M, n \in N$). Finally, it is easy to check that $P = MN$, where this semi-direct product MN is sensible since N is normal in P.

[6] *Trivial* G-action means that $g \cdot v = v$ for all $v \in V$.

The *longest Weyl element* of G is the matrix

$$w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The *Bruhat decomposition* of G, regardless of whether G is $\mathrm{GL}(2,k)$ or $\mathrm{SL}(2,k)$, is a disjoint union of double cosets PwN indexed by the Weyl element,

$$G = P \sqcup PwN.$$

Let $G = \mathrm{GL}(2,k)$. One containment is clear. To see that $G \subseteq P \sqcup PwN$, let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$. If $g \in P$, we are done. If $g \notin P$, then $c \neq 0$. Observe that for $\Delta = \det(g)$,

$$\begin{bmatrix} \Delta/c & a \\ 0 & c \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = g.$$

So if $g \notin P$, by the above display, $g \in PwN$. The computation for $G = \mathrm{SL}(2,k)$ is similar. Furthermore, PwN equals PwP (not just $PwN \subseteq PwP$), meaning G also decomposes as

$$G = P \sqcup PwP$$

which is especially useful in Chapter 3.

1.5 Characters of M, N, and P

Since M is abelian, by Complete Reducibility, M-representations decompose into characters. Therefore, we focus on M-characters. Our goal is to extend M-characters to P-characters and then induce them to G-characters. Since N is abelian, we could ideally do the same with N. However, as we shall see, the fact that M normalizes N allows M-characters to be extended to P-characters trivially. Lifting characters from N to characters of P, as it turns out, is less simple.

However, as suggested in a final counting argument of Chapter 3, extended characters from M and extended characters from N will each account for (very!) roughly half of the irreducible G-representations. Therefore, even though lifting from M is easier, doing so appears to be just as productive as the alternative.

Since characters of M need not be unique, we define an M-character here.

Definition 1.21. A *character* χ *of* M is the homomorphism $\chi : M \longrightarrow \mathbb{C}^\times$ such that for some characters χ_1, χ_2 of $k^\times$:

$$\text{if } G = \mathrm{GL}(2,k), \quad \chi\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) = \chi_1(a)\chi_2(d),$$

$$\text{if } G = \mathrm{SL}(2,k), \quad \chi\left(\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\right) = \chi_1(a).$$

We can now separate characters of M into two useful categories: regular and irregular.

Definition 1.22. For any character χ of M, define the related character

$$\chi^w : M \longrightarrow \mathbb{C}^\times \quad \chi^w(m) = \chi(wmw^{-1})$$

for long Weyl w. The character χ is *regular* if $\chi^w \neq \chi$ and is *irregular* otherwise.

For $G = \mathrm{GL}(2, k)$ and $m \in M$, we have that

$$wmw^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} d & 0 \\ 0 & a \end{bmatrix}.$$

Thus, $\chi^w(m) = \chi_1(d)\chi_2(a)$. Since χ is a homomorphism, it easily follows that χ is an irregular M-character if and only if $\chi_1 = \chi_2$. Equivalently, χ is an irregular M-character if and only if $\chi = \chi_0 \circ \det$, where χ_0 is some character of $k^\times$.

As promised, since M normalizes N, we may extend characters of M to characters of P by giving them trivial N-action. Namely, recall that $P = MN$ and define

$$\chi(p) = \chi(m) \text{ for } p = mn \in P.$$

This extended character is still a character (that is, is still a homomorphism). To see this, note that for $p_1 = m_1 n_1$ and $p_2 = m_2 n_2$, $p_1 p_2 = m_1 n_1 m_2 n_2 = m_1(m_2 m_2^{-1}) n_1 m_2 n_2$. Since M normalizes N, $m_2^{-1} n m_2 = \tilde{n}$ for some $\tilde{n} \in N$. Then

$$\chi(p_1 p_2) = \chi(m_1 m_2 \tilde{n} n_2) = \chi(m_1 m_2) = \chi(m_1)\chi(m_2) = \chi(p_1)\chi(p_2).$$

We can thus view χ as a P-representation with space $\mathbb{C}$ and P-action

$$p \cdot v = \chi(p)v \quad \text{ where } p \in P, v \in \mathbb{C}.$$

The next section will introduce induction and restriction functors. The induction functors will allow us to extend χ as a P-representation to a G-representation.

1.6 Induction and Restriction Functors

Let G be a finite group and H be a subgroup of G. Let A be an H-representation and B be a G-representation. We can view every B as an H-representation (and similarly, every G-linear map as an H-linear map) simply by forgetting some of the full G-action—that is, by restricting B to have H-action. Thus, we have a *restriction functor* or *forgetful functor*

$$\mathrm{Res}_H^G() : \{G\text{-representations}\} \longrightarrow \{H\text{-representations}\},$$
$$\mathrm{Res}_H^G() : \{G\text{-linear maps}\} \longrightarrow \{H\text{-linear maps}\}.$$

In the category of vector spaces, the restriction functor does nothing in the sense that for all G-representations B and G-linear maps f,

$$\begin{aligned} \mathrm{Res}_H^G(B) &= B \quad \text{as a vector space} \\ \mathrm{Res}_H^G(f) &= f \quad \text{as an linear map.} \end{aligned}$$

However, $\mathrm{Res}_H^G(B)$ *does not* fully equal B as a representation since restricting allowable group action changes its category (and similarly for $\mathrm{Res}_H^G(f)$). At the surface level, there seems to be no point to the restriction functor. However, restriction allows us to better describe the more interesting *right-adjoint induction functor*,

$$\begin{aligned} \mathrm{Ind}_H^G() &: \{H\text{-representations}\} \longrightarrow \{G\text{-representations}\}, \\ \mathrm{Ind}_H^G() &: \{H\text{-linear maps}\} \longrightarrow \{G\text{-linear maps}\}. \end{aligned}$$

Given an H-representation A, let the G-representation $\mathrm{Ind}_H^G(A)$ have as its representation space

$$\{f : G \longrightarrow A | f(hg) = h \cdot f(g) \text{ for all } h \in H, g \in G\}$$

with G-action

$$(g \cdot f)(x) = f(xg) \text{ for all } x, g \in G.$$

Furthermore, given an H-linear map $\varphi : A \longrightarrow A'$ (where A, A' are H-representations), we have the map

$$\mathrm{Ind}_H^G(\varphi) : \mathrm{Ind}_H^G(A) \longrightarrow \mathrm{Ind}_H^G(A') \quad f \longmapsto \varphi \circ f.$$

Note that $\varphi \circ f : G \longrightarrow A'$. Since φ, f are both H-maps, $\varphi \circ f$ is an H-map as well; so $\varphi \circ f \in \mathrm{Ind}_H^G(A')$. We wish to confirm that $\mathrm{Ind}_H^G(\varphi)$ is indeed G-linear. Clearly $\varphi \circ f$ is linear. So, observe that for $g, x \in G$ and $f \in \mathrm{Ind}_H^G(A)$,

$$\begin{aligned} (\mathrm{Ind}_H^G(\varphi)(g \cdot f))(x) &= (\varphi \circ (g \cdot f))(x) = \varphi((g \cdot f)(x)) && [\text{def. of } \mathrm{Ind}_H^G(\varphi)] \\ &= \varphi(f(xg)) = (\varphi \circ f)(xg) && [\text{def. of } G\text{-action}] \\ &= (g \cdot (\varphi \circ f))(x) && [\text{def. of } G\text{-action}] \\ &= (g \cdot \mathrm{Ind}_H^G(\varphi))(x) && [\text{def. of } \mathrm{Ind}_H^G(\varphi).] \end{aligned}$$

Hence, $\mathrm{Ind}_H^G()$ takes H-maps to G-maps. We call this the right-adjoint induction functor, because it is *right-adjoint to restriction.* That is, as proved carefully in Appendix A,

$$\mathrm{Hom}_H(\mathrm{Res}_H^G(B), A) \approx \mathrm{Hom}_G(B, \mathrm{Ind}_H^G(A)).$$

Note that an alternative *left-adjoint restriction functor* could have been used as well; the induced representation space would have been defined in terms of tensor products of spaces rather than spaces of functions. As it turns out, these functors are equivalent for finite groups, so there is no harm in choosing one over the other. In the next section, we examine a specific instance of inducing a representation.

1.7 Principal Series

We return to the environment of $G = \mathrm{GL}(2,k)$ or $G = \mathrm{SL}(2,k)$. Recall the groups M, N, and P. Let χ be a character of M extended trivially to P and thus viewed as a (one-dimensional) P-representation.

Definition 1.23. The *principal series* representation I_χ of G is $\mathrm{Ind}_P^G(\chi)$; that is, the principal series has the representation space

$$\{f : G \longrightarrow \mathbb{C} : f(px) = \chi(p)f(x) \;\text{ for all } p \in P,\; x \in G\}$$

with G-action

$$(g \cdot f)(x) = f(xg) \text{ for all } g, x \in G.$$

We mentioned that Frobenius reciprocity yields the isomorphism (proved in Appendix A)

$$\mathrm{Hom}_G(V, \mathrm{Ind}_P^G(W)) \approx \mathrm{Hom}_P(\mathrm{Res}_P^G(V), W)$$

for any G-representation V and P-representation W. Letting $W = \chi$ for our case gives

$$\mathrm{Hom}_G(V, I_\chi) \approx \mathrm{Hom}_P(\mathrm{Res}_P^G(V), \chi).$$

This isomorphism will let us later replace $\mathrm{Res}_P^G(V)$ with a quotient of it called the *Jacquet module* (Proposition 2.5). Heuristically, by making the right-side of the isomorphism have homomorphisms from a smaller domain, we make the left-side more understandable, which will lead to the main embedding result of Chapter 2.

1.8 Factoring a Function

This final introductory section addresses what it means for a function to *factor through* a quotient. We will show that, given a homomorphism φ between groups G and H, φ factors through a quotient G/N if and only if N is contained in the kernel of φ. This result is frequently invoked throughout the thesis. However, a reader well-versed in category theory should feel free to skip it.

Definition 1.24. Let the group homomorphism $\varphi : G \longrightarrow H$ be given, and let $\pi : G \longrightarrow \tilde{G}$ be a projection mapping. We say φ *factors through* the group $\tilde{G}$ if there exists some homomorphism $\varphi_0 : \tilde{G} \longrightarrow H$ such that $\varphi = \varphi_0 \circ \pi$. That is, we have the following commutative diagram.

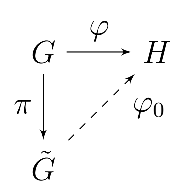


Figure 1.1: The map φ factors through $\tilde{G}$.

The phrase "factors" comes from the fact that $\varphi = \varphi_0 \circ \pi$, so it looks as if we have "factored" φ using the projection map π. Given this definition, we have the following proposition.

Proposition 1.25. Let the group homomorphism $\varphi : G \longrightarrow H$ be given, and let G/N be a quotient space. Then φ factors through the quotient map $\pi : G \longrightarrow G/N$ if and only if $N \subseteq \ker(\varphi)$.

Proof. ($\Rightarrow$) Suppose φ factors through $\pi : G \longrightarrow G/N$. Then, $\varphi = \varphi_0 \circ \pi$ for some $\varphi_0 : G/N \longrightarrow H$. So for $n \in N$,

$$\varphi(n) = \varphi_0(\pi(n)) = \varphi_0(1_{\tilde{G}}) = 1_H$$

so $n \in \ker(\varphi)$. Hence, $N \subseteq \ker(\varphi)$.

($\Leftarrow$) Let $N \subseteq \ker(\varphi)$. Then the natural map $\pi_\varphi : G/N \longrightarrow G/\ker(\varphi)$ is well-defined since $N \subseteq \ker(\varphi)$. By the first isomorphism theorem, $G/\ker(\varphi) = \mathrm{Im}(\varphi)$ by the bijection f that sends $g + \ker(\varphi)$ to $\varphi(g)$. Let $\iota : \mathrm{Im}(\varphi) \longrightarrow H$ be an inclusion.

Finally, let $\varphi_0 = \iota \circ f \circ \pi_\varphi : G/N \longrightarrow H$. Then for $g \in G$,

$$\varphi_0(g + N) = (\iota \circ f \circ \pi_\varphi)(g + N) = \iota(f(g + \ker(\varphi)) = \iota(\varphi(g)) = \varphi(g).$$

Therefore,

$$\varphi_0(\pi(g)) = \varphi_0(g + N) = \varphi(g)$$

meaning $\varphi = \varphi_0 \circ \pi$. Thus, the mapping property is satisfied. $\square$

Chapter 2

Two Embedding Results

This chapter establishes two embedding results involving principal series and the Jacquet module. The first states that an irreducible representation with a nontrivial Jacquet module is isomorphic to a nontrivial subrepresentation of some principal series I_χ; that is, the representation *embeds* in some I_χ. The second result, a dual to the first, states that irreducible quotients (images) of principal series have nontrivial Jacquet modules. Coupled together, these embedding results lead to the study of *intertwining operators.*

We begin by introducing the language of isotypes and co-isotypes. The characteristic mapping property of the co-isotype allows us to show that the Jacquet module, initially an N-representation, is also a P-representation. With the Jacquet module, Frobenius reciprocity yields a corollary isomorphism (Proposition 2.5), which in turn leads to the first desired embedding result (Theorem 2.6). Afterwards, we define a *dual* or *contragredient* representation. Equipped with this machinery, the second embedding result (Theorem 2.11) is proved.

2.1 Isotypes and Co-Isotypes

Let H be a finite group. Let V be a finite-dimensional representation of H, and let σ be an irreducible H-representation. Given our category, *every arrow appearing in this section is an H-map.*

Definition 2.1. The *σ-isotype* $V^\sigma \overset{\iota}{\hookrightarrow} V$ is the smallest subrepresentation V^σ of V with the following mapping property: for every $\varphi : \sigma \longrightarrow V$, there exists a unique $\varphi_0 : \sigma \longrightarrow V^\sigma$ such that $\varphi = \iota \circ \varphi_0$, where ι is simply an inclusion mapping. That is, we have the following commutative diagram.

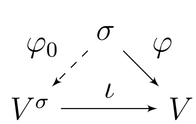


Figure 2.1: Mapping property of the σ-isotype, V^σ.

This definition *characterizes* V^σ. We can confirm that V^σ exists by *constructing* it as

$$V^\sigma = \sum_{\varphi:\sigma\longrightarrow V} \mathrm{Im}(\varphi),$$

as done in Appendix B. However, in practice, we will only refer to V^σ by its characterization. Note that the homomorphic image of an irreducible representation σ is either 0 or an irreducible representation isomorphic to σ. Thus, we see that V^σ comprises all copies of σ in V.

In a similar vein, we can define the σ-co-isotype.

Definition 2.2. The *σ-co-isotype* $V \overset{\pi}{\twoheadrightarrow} V_\sigma$ is the smallest quotient representation V_σ of V with the following mapping property: for every $\varphi : V \longrightarrow \sigma$, there exists a unique $\varphi_0 : V_\sigma \longrightarrow \sigma$ such that $\varphi = \varphi_0 \circ \pi$. Thus, we have the following commutative diagram.

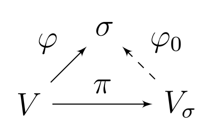


Figure 2.2: Mapping property of the σ-co-isotype, V_σ.

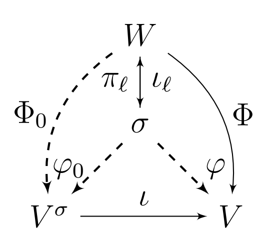
We can show that the σ-co-isotype exists by constructing it as

$$V_\sigma = V/K \quad K = \bigcap_{\varphi\in\mathrm{Hom}_H(V,\sigma)} \ker(\varphi).$$

This construction is also confirmed in Appendix B. We now address the functorial aspects of the isotype and co-isotype. Let W be any finite product/coproduct of H-representations each isomorphic to σ, and let Λ be an index set for the product. Suppose we are given a map $\Phi : W \longrightarrow V$.

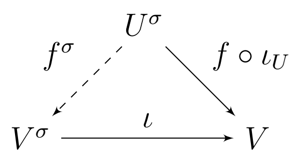
For every index $\ell \in \Lambda$, we have the injection $\iota_\ell : \sigma \longrightarrow W$. Then $\Phi \circ \iota_\ell \overset{\text{call}}{=} \varphi : \sigma \longrightarrow V$, so by the isotype mapping property, there exists a unique $\varphi_0 : \sigma \longrightarrow V^\sigma$ such that $\varphi = \iota \circ \varphi_0$, where $\iota : V^\sigma \longrightarrow V$ is the inclusion mapping.

Furthermore, we have projection mappings $\pi_\ell : W \longrightarrow \sigma$. Then, by letting $\Phi_0 = \varphi_0 \circ \pi_\ell : W \longrightarrow V^\sigma$, Φ_0 is unique since it is the composition of two unique maps. Hence, we have a commutative diagram below.

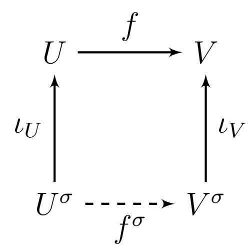


So for every $\Phi : W \longrightarrow V$, there exists a unique $\Phi_0 : W \longrightarrow V^\sigma$ such that $\Phi = \iota \circ \Phi_0$. Thus, we see that W has the mapping property of the isotype, meaning that we can replace σ with any finite product/coproduct of copies of σ and retain the mapping property. The same generalization easily holds for the σ-co-isotype; just reverse all of the arrows in question.

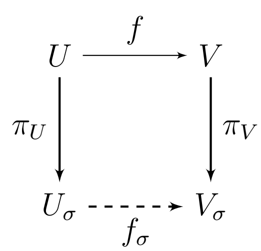
This *modified* mapping property can then be used to show that the isotype and co-isotype are functorial. Namely, given a map $f : U \longrightarrow V$, where U, V are finite-dimensional representations, we have the inclusion mappings $\iota_U : U^\sigma \longrightarrow U$ and $\iota_V : V^\sigma \longrightarrow V$. By its construction, U^σ is a finite product of copies of σ (because sums and products of subspaces are isomorphic in finite-dimensional cases). Since $f \circ \iota_U : U^\sigma \longrightarrow V$ is given, by the modified isotype mapping property, there exits a unique $f^\sigma : U^\sigma \longrightarrow V^\sigma$ such that the diagram below commutes.



Intuitively, we are saying that given a map $f : U \longrightarrow V$, we have maps between the constituent *parts* of U and V — that is, between finite products/coproducts of single, irreducible subrepresentations within each. Equivalently, we have the following diagram.



In a similar manner, given the σ-co-isotype and a map $f : U \longrightarrow V$, we have a map between U_σ and V_σ.



The modified mapping property of the co-isotype is specifically used in the next section.

2.2 The Jacquet Module

Recall the matrix groups G, M, N, and P. Let 1_N denote the trivial N-representation; that is, 1_N is a copy of $\mathbb{C}$ with trivial N-action.

Definition 2.3. The *Jacquet module* V_N of a complex G-representation V is

$$V_N = \mathrm{Res}_N^G(V)/K \qquad K = \bigcap_{\varphi \in \mathrm{Hom}_N(\mathrm{Res}_N^G(V), 1_N)} \ker \varphi.$$

Note that V_N is the 1_N-co-isotype of $\mathrm{Res}_N^G(V)$. First, we show that K is N-stable so that V_N is indeed an N-representation. Let $k \in K$, so $\varphi(k) = 0$ for all $\varphi \in \mathrm{Hom}_N(\mathrm{Res}_N^G(V), 1_N)$. Then for $n \in N$, $\varphi(n \cdot k) = n \cdot \varphi(k) = 0$, since φ is an N-map. Thus, $n \cdot k \in K$, so K is N-stable. In general for $v \in \mathrm{Res}_N^G(V)$,

$$\varphi(n \cdot v) = n \cdot \varphi(v) = \varphi(v)$$

since $\varphi(v) \in 1_N$. By linearity, $\varphi(n \cdot v - v) = 0$, so $n \cdot v - v \in K$ since φ was arbitrary. Furthermore, the N-action on V_N is trivial since for $v \in \mathrm{Res}_N^G(V)$,

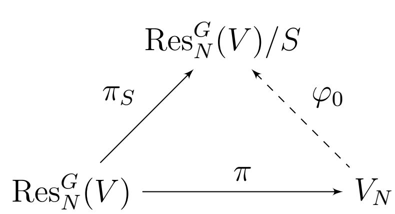
$$\begin{aligned} n \cdot (v + K) &= n \cdot v + K && \text{[def. of quotient action]} \\ &= v + K && \text{[since } n \cdot v - v \in K.] \end{aligned}$$

Let $S = \mathrm{span}\{n \cdot v - v : v \in \mathrm{Res}_N^G(V), n \in N\}$.[1] Since $n \cdot v - v \in K$ and K is a complex vector space, $S \subseteq K$. The opposite containment follows from the mapping property of the co-isotype. Specifically, first note that S is N-stable, so $\mathrm{Res}_N^G(V)/S$ is an N-representation. Furthermore, the N-action on $\mathrm{Res}_N^G(V)/S$ is trivial, since $n \cdot v - v \in S$ implies that

$$n \cdot (v + S) = n \cdot v + S = v + S.$$

So, since $\mathrm{Res}_N^G(V)/S$ is an N-representation with *trivial* N-action for abelian N, by Corollary 1.20, $\mathrm{Res}_N^G(V)/S = \bigoplus 1_N$.

Also, we are given the quotient map $\pi_S : \mathrm{Res}_N^G(V) \longrightarrow \mathrm{Res}_N^G(V)/S$. From the *modified* co-isotype mapping property for finite product $\bigoplus 1_N$ (in finite-dimensions, direct sums and direct products are isomorphic), we have the following commutative diagram.



Thus, the quotient map π_S factors through $V_N = \mathrm{Res}_N^G(V)/K$, so by Section 1.8, $K \subseteq S$. By mutual containment, K now has two descriptions:

$$K = \bigcap_{\varphi \in \mathrm{Hom}_N(\mathrm{Res}_N^G(V), 1_N)} \ker \varphi = \mathrm{span}\{n \cdot v - v : v \in \mathrm{Res}_N^G(V), n \in N\}.$$

[1] By *span*, we mean *algebraic span*, or finite linear combinations of $(n \cdot v - v)$ with complex coefficients.

The Jacquet module V_N has trivial N-action, so we want to extend this action (nontrivially) to P by defining an M-action. We use (of course) the fact that M normalizes N. We know that M-action *descends* from V to $V_N = \mathrm{Res}_N^G / K$ provided that K is M-stable. For any $m \in M$ and $\varphi \in \mathrm{Hom}_N(\mathrm{Res}_N^G(V), 1_N)$, consider the map

$$\varphi \longmapsto \varphi_m, \qquad \text{where} \qquad \varphi_m : \mathrm{Res}_N^G(V) \longrightarrow 1_N, \quad \varphi_m(v) = \varphi(m \cdot v).$$

Then φ_m is an N-homomorphism. To see this, observe that

$$\varphi_m(n \cdot v) = \varphi(m \cdot (n \cdot v)) = \varphi((mn) \cdot v) = \varphi(n'm \cdot v),$$

where the last equality follows from M normalizing N. Since φ is an N-map,

$$\varphi(n'm \cdot v) = \varphi(n' \cdot (m \cdot v)) = n' \cdot \varphi(m \cdot v) = \varphi(m \cdot v) = \varphi_m(v),$$

where the second-to-last equality follows from $\varphi(m \cdot v) \in 1_N$. Overall, φ_m is again an N-homomorphism. The punch-line is that φ_m kills N-action since φ already does so and M normalizes N.

To finally show that K is M-stable, we need to note that the map $\varphi \longmapsto \varphi_m$ simply permutes $\mathrm{Hom}_N(\mathrm{Res}_N^G(V), 1_N)$. The next lemma quickly confirms this fact.

Lemma 2.4. The map $\varphi \longmapsto \varphi_m$ is a permutation of $\mathrm{Hom}_N(\mathrm{Res}_N^G(V), 1_N)$.

Proof. Consider the map $\varphi \longmapsto \varphi_{m^{-1}}$ in $\mathrm{Hom}_N(\mathrm{Res}_N^G(V), 1_N)$ Then composing $(\varphi \longmapsto \varphi_m)$ with $(\varphi \longmapsto \varphi_{m^{-1}})$ in either order yields the identity, so $\varphi \longmapsto \varphi_m$ has an inverse and is thus bijective. □

Now varying φ through all $\mathrm{Hom}_N(\mathrm{Res}_N^G(V), 1_N)$ yields

$$\begin{aligned} k \in K &\Leftrightarrow k \in \ker \varphi && \text{[def. of } K] \\ &\Leftrightarrow k \in \ker \varphi_m && [\text{since } \varphi \longmapsto \varphi_m \text{ is a perm.}] \\ &\Leftrightarrow m \cdot k \in \ker \varphi && [\text{def. of } \varphi_m] \\ &\Leftrightarrow m \cdot k \in K. \end{aligned}$$

Thus, $m \cdot K \subseteq K$ (in fact, $m \cdot K = K$) for arbitrary $m \in M$, meaning K is M-stable. Hence, V_N is an M-representation.

Since K is M-stable and N-stable, K is clearly P-stable. Thus, V_N is also a P-representation with action

$$p \cdot (v + K) = m \cdot v + K \qquad (v \in \mathrm{Res}_N^G(V), p = mn \in P).$$

By using the Jacquet module V_N as a P-representation, we can finally establish the first embedding result of this chapter.

2.3 First Embedding Result

To start, we have a corollary to Frobenius reciprocity. Note that $\mathrm{Hom}_P(V_N, \chi)$ is now sensible since V_N is a P-representation.

Proposition 2.5. (Corollary Isomorphism). For any character χ of M and representation V of G,

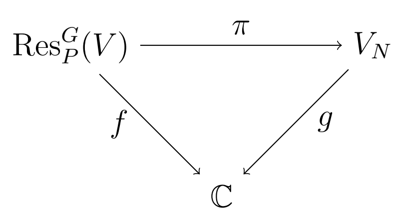
$$\mathrm{Hom}_G(V, I_\chi) \approx \mathrm{Hom}_P(V_N, \chi),$$

where χ of course is understood to be a P-representation extended from M.

Proof. By Frobenius reciprocity, we just need to show that

$$\mathrm{Hom}_P(\mathrm{Res}^G_P(V), \chi) \approx \mathrm{Hom}_P(V_N, \chi).$$

Recall that the representation space of χ is $\mathbb{C}$. Suppose we are given a linear P-map $g : V_N \longrightarrow \mathbb{C}$. Then, for quotient map $\pi : \mathrm{Res}^G_P(V) \longrightarrow V_N$, consider the *provisional* diagram below.



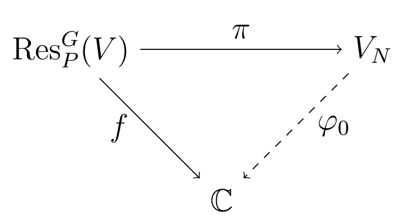
We need every $f \in \mathrm{Hom}_P(\mathrm{Res}^G_P(V), \chi)$ to map bijectively to some $g \in \mathrm{Hom}_P(V_N, \chi)$. From the diagram, a map in one direction is immediate; that is,

$$\mathrm{Hom}_P(V_N, \chi) \longrightarrow \mathrm{Hom}_P(\mathrm{Res}^G_N(V), I_\chi), \quad g \longmapsto g \circ \pi.$$

It is easily verified that $g \circ \pi$ is a P-map. On the other hand, suppose we are given the map $f \in \mathrm{Hom}_P(\mathrm{Res}^G_P(V), \chi)$. Then for $p = mn \in P$,

$$f(p \cdot v) = \chi(p)f(v) = \chi(mn)f(v) = \chi(m)f(v).$$

In particular, $f(n \cdot v) = f(v)$, so $f \in \mathrm{Hom}_N(\mathrm{Res}^G_N(V), 1_N)$. Then, $f(K) = 0$, so $K \subseteq \ker(f)$. By Section 1.8, f factors through $\mathrm{Res}^G_P(V)/K = V_N$. That is, we have an induced map $\varphi_0 : V_N \longrightarrow \mathbb{C}$.



Also, the fact that φ_0 is a P-map follows easily from f being a P-map. Thus, $\varphi_0 \in \mathrm{Hom}_P(V_N, \chi)$. Then, we have a map $\mathrm{Hom}_P(\mathrm{Res}^G_P(V), \chi) \longrightarrow \mathrm{Hom}_P(V_N, \chi)$

by $f \longmapsto \varphi_0$, where φ_0 is understood to be the induced map above. Hence, for $f \in \mathrm{Hom}_P(\mathrm{Res}_P^G(V), \chi)$, we have that the composition

$$(g \mapsto g \circ \pi) \circ (f \mapsto \varphi_0) = \varphi_0 \circ \pi = f$$

is the identity map on $\mathrm{Hom}_P(\mathrm{Res}_P^G(V), \chi)$. Similarly, the reverse composition is the identity on $\mathrm{Hom}_P(V_N, \chi)$, so we have found a bijection. □

The Corollary Isomorphism is then used in the embedding result below.

Theorem 2.6. *Let V be an irreducible, finite-dimensional representation of G. Then, V is isomorphic to a nontrivial subrepresentation of some principal series I_χ if and only if V_N is nontrivial.*

Proof. ($\Rightarrow$) Suppose for contraposition that V_N is trivial, so $V_N = 0$. By the Corollary Isomorphism, $\mathrm{Hom}_G(V, I_\chi) \approx \mathrm{Hom}_P(0, \chi)$, so $\mathrm{Hom}_G(V, I_\chi)$ and $\mathrm{Hom}_P(0, \chi)$ each only contain the zero-map. Hence, there is no nontrivial G-map from V to any principal series I_χ, so V cannot be isomorphic to a nontrivial subrepresentation of I_χ.

($\Leftarrow$) Suppose $V_N \neq 0$. If V_N is not irreducible, then V_N has a proper irreducible subrepresentation and hence a proper quotient representation. If this proper quotient is not irreducible, then this quotient has a proper quotient representation for the same reason. This process must terminate by finite-dimensionality, yielding (since a quotient of a quotient is a quotient) an irreducible quotient representation χ of V_N. Then, we have the quotient map

$$V_N \longrightarrow \chi.$$

Note that the P-action on V_N is abelian since M is abelian, M normalizes N, and the N-action on V_N is trivial. That is, for $v \in V_N$ and $p, p' \in P$,

$$pp' \cdot v = (mn)(m'n') \cdot v = mnm' \cdot v = m(m'\tilde{n}) \cdot v = mm' \cdot v$$

and since $mm' = m'm$, reversing the steps yields the desired result. Since the P-action is abelian, by Theorem 1.18, irreducible P-representations of V_N are one-dimensional. Thus, χ is a character. Hence, there is a nonzero P-homomorphism $V_N \longrightarrow \chi$.

By the Corollary Isomorphism, there is a nonzero G-homomorphism $f : V \longrightarrow I_\chi$. Since V is irreducible and f is nonzero, f is injective (for otherwise its kernel would be a proper, nontrivial subrepresentation of V). Thus, V is isomorphic to a nontrivial subrepresentation of I_χ—specifically, the nontrivial subrepresentation is $\mathrm{Im}(f)$. □

2.4 Contragredient Representations

Throughout this section, let H be any finite group. Recall that a *linear functional* is a linear map from a vector space V to its field of scalars.

Definition 2.7. Let V be a vector space. Then its *dual space* V^* is the set of all linear functionals on V.

Definition 2.8. Let V be an H-representation. Then its *contragredient* or *dual representation* is the dual space V^* with H-action defined by

$$(h \cdot \lambda)(v) = \lambda(h^{-1} \cdot v) \qquad \text{for } \lambda \in V^*, h \in H, v \in V.$$

The "h^{-1}" allows the action to be associative. Now, we return to the working environment $G = \mathrm{GL}(2, k)$ (or $G = \mathrm{SL}(2, k)$). With these "dual" properties in play, we can show that the dual space of the Jacquet module is the Jacquet module of the dual space.

Proposition 2.9. (Contragredient commutes with Jacquet). Let V be a finite-dimensional representation of G. Then,

$$(V_N)^* \approx (V^*)_N.$$

Proof. Let W be any G-representation of V with Jacquet module W_N. For ease of notation throughout this proof, we write $W_N = W/S_W$ rather than $W_N = \mathrm{Res}_N^G(W)/S_W$ with the understanding that W_N is treated as an N-representation; in a similar vein, we write $S_W = \mathrm{span}\{n \cdot w - w : w \in W, n \in N\}$ rather than the same span over $w \in \mathrm{Res}_N^G(W)$.

Define π_W to be the quotient map from W to W_N, so π_W takes G-representations to N-representations. That is,

$$\pi_W : W \longrightarrow W_N \quad w \longmapsto w + S_W.$$

Then for $\pi_V : V \longrightarrow V_N$, consider its pullback mapping,

$$(\pi_V)^* : (V_N)^* \longrightarrow V^*, \quad \lambda \longmapsto \lambda \circ \pi_V.$$

From linear algebra [3], the pullback of a surjection is an injection. So, $(\pi_V)^*$ is an injection. Thus, $(\pi_V)^*(\lambda)$ is the zero map on V if and only if λ is the zero map on V_N. We now compute that $(\pi_V)^*(\lambda) \in V^*$ has trivial N-action. Observe that for $v \in V$,

$$\begin{aligned}
(n \cdot (\pi_V)^*\lambda)(v) &= ((\pi_V)^*\lambda)(n^{-1} \cdot v) && \text{[def. of dual action]}\\
&= (\lambda \circ \pi_V)(n^{-1} \cdot v) && \text{[def. of pullback]}\\
&= (\lambda \circ \pi_V)(v) && [n^{-1} \cdot v - v \in S_V]\\
&= ((\pi_V)^*\lambda)(v) && \text{[def. of pullback.]}
\end{aligned}$$

We wish to show that

$$(V_N)^* \overset{(\pi_V)^*}{\longrightarrow} V^* \overset{\pi_{V^*}}{\longrightarrow} (V^*)_N$$

is a bijection. First, we show that the composition is an injection. Note that

$$\pi_{V^*} : V^* \longrightarrow (V^*)_N \quad v \longmapsto v + S_{V^*}$$

for $S_{V^*} = \mathrm{span}\{n \cdot v - v : v \in V^*, n \in N\}$. Then, letting $|N|$ denote the measure of (finite) N, we have $\frac{1}{|N|}\int_N dn = 1$. So for $v \in V^*$,

$$v - \frac{1}{|N|}\int_N n \cdot v \; dn = \frac{v}{|N|}\int_N \; dn - \frac{1}{|N|}\int_N n \cdot v \; dn = \frac{1}{|N|}\int_N (v - n \cdot v) \; dn \in S_{V^*}$$

Thus, we have equivalent cosets $v + S_{V^*} = (\frac{1}{|N|}\int_N n \cdot v \, dn) + S_{V^*}$. Then π_{V^*} is now

$$V^* \longrightarrow (V^*)_N, \quad v \longmapsto \int_N n \cdot v \, dn + S_{V^*}$$

Let $\mu \in V^*$ be N-fixed. Then,

$$\mu = \frac{\mu}{|N|}\int_N dn = \frac{1}{|N|}\int_N \mu \, dn = \frac{1}{|N|}\int_N n \cdot \mu \, dn.$$

By [C.2] in Appendix C, N-fixed μ is in the kernel of π_{V^*} if and only if $\int_N n \cdot \mu \, dn = 0$. Hence, by the calculation above,

$$\mu \in \ker(\pi_{V^*}) \Leftrightarrow \int_N n \cdot \mu \, dn = 0 \Leftrightarrow \mu = 0.$$

Since we showed that $(\pi_V)^*(\lambda)$ is N-fixed, we have that $(\pi_V)^*(\lambda) \in \ker(\pi_{V^*})$ if and only if $(\pi_V)^*(\lambda) = 0$.

Now suppose λ is in the kernel of the composition $\pi_{V^*} \circ (\pi_V)^* : (V_N)^* \longrightarrow (V^*)_N$, so $\pi_{V^*}((\pi_V)^*(\lambda)) = 0$. Then $(\pi_V)^*(\lambda) \in \ker(\pi_{V^*})$, so $(\pi_V)^*(\lambda) = 0$. However, since $(\pi_V)^*$ is an injection, $\lambda = 0$. Thus, the whole composition is injective.

Similarly, we know that the quotient map $\pi_{V^*} : V^* \longrightarrow (V^*)_N$ is surjective, so its pullback $((V^*)_N)^* \longrightarrow V^{**}$ is injective. Then, by repeating the argument above, we have that

$$((V^*)_N)^* \longrightarrow V^{**} \longrightarrow (V^{**})_N$$

is injective. However, since V is finite-dimensional, $V^{**} \approx V$. Thus, we have an injection from $((V^*)_N)^*$ to V_N. Given finite-dimensionality, its pullback $(V_N)^* \longrightarrow ((V^*)_N)^{**} = (V^*)_N$ surjects [3]. Hence, we have a bijection between $(V_N)^*$ and $(V^*)_N$, as desired. □

Thus, we have the following commutative diagram in which the distinction between $(V_N)^*$ and $(V^*)_N$ is suppressed.

$$\begin{array}{ccc} V & \xrightarrow{\;*\;} & V^* \\ \downarrow{\scriptstyle \pi_V} & & \downarrow{\scriptstyle \pi_{V^*}} \\ V_N & \xrightarrow{\;*\;} & V_N^* \end{array}$$

Proposition 2.10. (Dual commutes with induction). Let K be a subgroup of H, and let σ be a K-representation. Then,

$$\mathrm{Ind}_K^H(\sigma^*) \approx (\mathrm{Ind}_K^H(\sigma))^*.$$

For a careful proof, see [10]. From Theorem 2.6, if an irreducible G-representation V has a nontrivial Jacquet module, then V embeds into some principal series. Now equipped with a contragredient functor that commutes with the Jacquet module and induction, we can establish a dual embedding result.

2.5 Second Embedding Result

Theorem 2.11. *Let V be an irreducible G-representation that is a quotient of a principal series I_χ. Then, the Jacquet module V_N is nontrivial.*

Proof. Let $I_\chi = \mathrm{Ind}_P^G(\chi)$ be some principal series, where χ of course is an M-character extended to P. Let V be a quotient of I_χ. Then there is a surjection

$$\mathrm{Ind}_P^G(\chi) \twoheadrightarrow V.$$

So, we dualize to get an injection

$$V^* \longrightarrow (\mathrm{Ind}_P^G(\chi))^* \approx \mathrm{Ind}_P^G(\chi^*),$$

where the isomorphism follows from the previous proposition. From the first embedding result (Theorem 2.6), $(V^*)_N \neq 0$. So, by Proposition 2.9, $(V_N)^* \neq 0$, so $V_N \neq 0$. Hence, the Jacquet module V_N is nontrivial. □

Chapter 3

Irreducible Principal Series and Hom-Group Decomposition

This chapter begins by introducing the notion of *intertwining operators* as a natural consequence of the two embedding results in Chapter 2. Given the apparent link between intertwining operators, principal series, and irreducible G-representations, we attempt to decompose the homomorphism group of intertwining operators between generic principal series. The decomposition will determine conditions for which a principal series I_χ is irreducible or comprises two irreducible subrepresentations; namely, these conditions will be based on the regularity or irregularity (Definition 1.22) of χ. With a rough counting argument (Section 3.4), we can show that the irreducibles found by examining I_χ account for roughly half of the q^2-1 irreducible representations of $\mathrm{GL}(2,k)$.

3.1 Intertwining Operators

Combining the embedding results (Theorems 2.6 and 2.11) immediately yields the following theorem.

Theorem 3.1. *If an irreducible G-representation V is a quotient of some principal series I_α, then V is isomorphic to a nontrivial subrepresentation of some I_β; that is, V embeds into some I_β.*

That is, if irreducible V is a quotient of I_α, we have a G-homomorphism called an *intertwining operator*,

$$I_\alpha \longrightarrow I_\beta$$

to some I_β.[1] In the general literature [4, 7, 8, 14] an intertwining operator is defined as any G-linear map between representations. Here we restrict the definition to mean a G-linear map specifically between principal series representations. Note that the map $I_\alpha \longrightarrow I_\beta$ is neither injective nor surjective.

We should now have somewhat of a sense as to why principal series representations matter; namely, an irreducible V that is a quotient of a principal series ends up being

[1]Mapping from I_α follows from the first isomorphism theorem of group theory.

between principal series. Accordingly, it seems natural to study the homomorphism group of all intertwining operators between principal series. Several of these intertwining operators will end up being so simple (such as zero-maps or scaling maps) that the irreducibles between them will be *forced*. That is, studying intertwining operators will eliminate several possibilities as to what an irreducible of $\mathrm{GL}(2,k)$ can be.

3.2 Decomposition Formula

Let A, B be subgroups of finite H. Let α be a character of A and β be a character of B. The goal of this section is to decompose the hom-group

$$\mathrm{Hom}_H(\mathrm{Ind}_A^H(\alpha), \mathrm{Ind}_B^H(\beta)).$$

By Frobenius reciprocity for finite groups (Appendix A),

$$\mathrm{Hom}_H(\mathrm{Ind}_A^H(\alpha), \mathrm{Ind}_B^H(\beta)) \approx \mathrm{Hom}_B(\mathrm{Res}_B^H(\mathrm{Ind}_A^H(\alpha)), \beta).$$

As a representation space,

$$\mathrm{Res}_H^B(\mathrm{Ind}_A^H(\alpha)) = \{f : H \longrightarrow \mathbb{C} : f(a \cdot h) = a \cdot f(h)\}$$

where the action on each f is restricted to right regular B-action. For any double coset $AhB \in A\backslash H/B$, define the projection operator

$$p_{AhB} : \mathrm{Res}_B^H(\mathrm{Ind}_A^H(\alpha)) \longrightarrow \mathrm{Res}_H^B(\mathrm{Ind}_A^H(\alpha), \quad (p_{AhB}f)(x) = \begin{cases} f(x) & \text{if } x \in AhB \\ 0 & \text{else.} \end{cases}$$

Also, p_{AhB} is a B-map since

$$\begin{aligned}
(p_{AhB}(b \cdot f))(x) &= \begin{cases} (b \cdot f)(x) & \text{if } x \in AhB \\ 0 & \text{else} \end{cases} && [\text{def. of } p_{AhB}] \\
&= \begin{cases} f(xb) & \text{if } x \in AhB \\ 0 & \text{else} \end{cases} && [\text{def. of right regular action}] \\
&= \begin{cases} f(xb) & \text{if } xb \in AhB \\ 0 & \text{else} \end{cases} \\
&= (p_{AhB}f)(xb) \\
&= b \cdot (p_{AhB}f)(x).
\end{aligned}$$

Note that the third equality follows from the fact that right-translating each element of AhB by some $b \in B$ simply permutes the coset. Since double cosets of any group partition that group, we have that for $x \in H$,

$$\sum_{AhB} (p_{AhB}f)(x) = f(x), \text{ so } \sum_{AhB} p_{AhB}(f) = f.$$

Thus, $\sum_{AhB} p_{AhB}$ is the identity on $\text{Res}_B^H(\text{Ind}_A^H(\alpha))$. That is, *in sum*,

$$\text{Res}_B^H(\text{Ind}_A^H(\alpha)) = \bigoplus_{AhB} p_{AhB}(\text{Res}_B^H(\text{Ind}_A^H(\alpha))).$$

Now, fix the double coset AhB. Then, the map

$$A\backslash AhB \longrightarrow (h^{-1}Ah \cap B)\backslash B, \quad Ahb \longmapsto (h^{-1}Ah \cap B)b$$

is a bijection, as proved by [C.3] of Appendix C. Let $\alpha^h : h^{-1}Ah \cap B \longrightarrow \mathbb{C}^\times$ such that $\alpha^h(b_0) = \alpha(hb_0h^{-1}) = \alpha(a)$ for $b_0 = h^{-1}ah$. Then the map

$$\begin{aligned} p_{AhB}(\text{Res}_B^H(\text{Ind}_A^H(\alpha))) &\longrightarrow \text{Ind}_{h^{-1}Ah\cap B}^B(\alpha^h) \\ f &\longmapsto \tilde{f} \quad \text{ where } \tilde{f}(b) = f(hb) \text{ for } b \in B, h \in H \end{aligned}$$

is a bijective B-map that follows from the previous bijection. Thus, we calculate that

$$\begin{aligned} \text{Hom}_H(\text{Ind}_A^H(\alpha), \text{Ind}_B^H(\beta)) &= \text{Hom}_B(\text{Res}_B^H(\text{Ind}_A^H(\alpha)), \beta) && (3.1) \\ &= \text{Hom}_B\Big(\bigoplus_{AhB\in A\backslash H/B} p_{AhB}(\text{Res}_B^H(\text{Ind}_A^H(\alpha))), \beta\Big) && (3.2) \\ &= \bigoplus_{AhB\in A\backslash H/B} \text{Hom}_B(p_{AhB}(\text{Res}_B^H(\text{Ind}_A^H(\alpha))), \beta) && (3.3) \\ &= \bigoplus_{AhB\in A\backslash H/B} \text{Hom}_B(\text{Ind}_{h^{-1}Ah\cap B}^B(\alpha^h), \beta), && (3.4) \end{aligned}$$

where (3.1) uses Frobenius, (3.2) uses $\sum_{AhB} p_{AhB} = \text{id}$, and (3.4) uses the bijection above. In general, for finite-dimensional vector spaces X, Y and group B, we have the *dual isomorphism*

$$\text{Hom}_B(X, Y) \approx \text{Hom}_B(Y^*, X^*)$$

as proved in [C.4] of Appendix C. Given this fact, the calculation continues.

$$\begin{aligned} \text{Hom}_H(\text{Ind}_A^H(\alpha), \text{Ind}_B^H(\beta)) &= \bigoplus_{AhB\in A\backslash H/B} \text{Hom}_B(\text{Ind}_{h^{-1}Ah\cap B}^B(\alpha^h), \beta) && (3.5) \\ &= \bigoplus_{AhB\in A\backslash H/B} \text{Hom}_B(\beta^{-1}, \text{Ind}_{h^{-1}Ah\cap B}^B(\alpha^h)^*) && (3.6) \\ &= \bigoplus_{AhB\in A\backslash H/B} \text{Hom}_B(\beta^{-1}, \text{Ind}_{h^{-1}Ah\cap B}^B((\alpha^h)^{-1})) && (3.7) \\ &= \bigoplus_{AhB\in A\backslash H/B} \text{Hom}_{h^{-1}Ah\cap B}(\text{Res}_{h^{-1}Ah\cap B}^B(\beta^{-1}), (\alpha^h)^{-1}) && (3.8) \\ &= \bigoplus_{AhB\in A\backslash H/B} \text{Hom}_{h^{-1}Ah\cap B}(\text{Res}_{h^{-1}Ah\cap B}^B(\beta)^*, (\alpha^h)^{-1}) && (3.9) \\ &= \boxed{\bigoplus_{AhB\in A\backslash H/B} \text{Hom}_{h^{-1}Ah\cap B}(\alpha^h, \text{Res}_{h^{-1}Ah\cap B}^B(\beta))}. && (3.10) \end{aligned}$$

Note that (3.6) uses the dual isomorphism, (3.7) uses "Contragredient commutes with induction" (Proposition 2.9), (3.8) uses Frobenius, (3.9) uses "Contragredient commutes with restriction" (which should be clear), and (3.10) uses the dual isomorphism again.

We now see how the contragredient and Frobenius work together to produce the calculation. Namely, Frobenius reciprocity only allows us to remove the induction functor from the *right-side* of the hom-group—that is, from its codomain. As a result, Frobenius only simplifies the right-side. However, by using the dual isomorphism, we can "switch" the right and left sides, simplify with Frobenius, and switch back; the fact that the contragredient commutes with everything in sight allows the switching to run smoothly.

Let us take a moment to appreciate the work done here. We have decomposed a hom-group between unknown principal series into a direct sum of hom-groups between one-dimensional characters α^h and β, which are well-understood (Corollary 1.17).

3.3 Consequences of Decomposition Formula

We now apply the totally general decomposition formula to $H = G$ and $A = B = P$. The next two corollaries capture how decomposition allows us to find representations of G related to an arbitrary principal series I_χ. As promised, the conditions for which these related representations are irreducible are contingent upon whether χ is regular or irregular.

Corollary 3.2. Let χ be a regular character of M (so $\chi^w \neq \chi$). Then $\mathrm{End}_G(I_\chi) = \mathbb{C}$; that is, the only G-maps from I_χ to I_χ are scaling maps. If χ is irregular (so $\chi^w = \chi$), then $\mathrm{End}_G(I_\chi) = \mathbb{C} \oplus \mathbb{C}$.

Proof. By definition, $\mathrm{End}_G(I_\chi) = \mathrm{Hom}_G(I_\chi, I_\chi) = \mathrm{Hom}_G(\mathrm{Ind}_P^G(\chi), \mathrm{Ind}_P^G(\chi))$, where we have extended χ to be a character of P with trivial N-action. Substituting from the boxed formula,

$$\mathrm{End}_G(I_\chi) = \bigoplus_{PgP \in P\backslash G/P} \mathrm{Hom}_{g^{-1}Pg\cap P}(\chi^g, \mathrm{Res}^P_{g^{-1}Pg\cap P}(\chi)).$$

Recall that $G = P \sqcup PwN = P \sqcup PwP$. Hence, the direct sum above is only indexing *two double cosets*: P (which equals $P1_GP$) and PwP. Thus,

$$\mathrm{End}_G(I_\chi) = \mathrm{Hom}_P(\chi^1, \chi) \;\oplus\; \mathrm{Hom}_{w^{-1}Pw\cap P}(\chi^w, \mathrm{Res}^P_{w^{-1}Pw\cap P}(\chi)).$$

By Corollary 1.17, if $\chi^w \neq \chi$ then $\mathrm{Hom}_{w^{-1}Pw\cap P}(\chi^w, \mathrm{Res}_{w^{-1}Pw\cap P}(\chi)) = \{0\}$, and if $\chi^w = \chi$, then $\mathrm{Hom}_{w^{-1}Pw\cap P}(\chi^w, \mathrm{Res}_{w^{-1}Pw\cap P}(\chi)) = \mathbb{C}$. By Schur's Lemma, $\mathrm{Hom}_P(\chi^1, \chi)$ $= \mathbb{C}$. So by definition of regular and irregular, the conclusion follows. □

Corollary 3.3. Let χ be a character of M. If χ is regular, then I_χ is irreducible and $I_\chi \approx I_{\chi^w}$. If $G = \mathrm{GL}(2, k)$ and χ is irregular, then χ is a character of G and $I_\chi = \chi \oplus W$ for some irreducible G-representation W.

Proof. First suppose χ is regular. If I_χ has a G-subrepresentation U, then from Chapter 1 we know that $I_\chi = U \oplus W$ for a G-representation W. Consider the projection mapping $\pi : I_\chi \longrightarrow U$. Since $\mathrm{End}_G(I_\chi) = \mathbb{C}$, π must be multiplication by a scalar—for otherwise we could define a non-scaling G-endomorphism on I_χ. If π is the zero-map, $U = 0$. If π is nonzero-scaling, $I_\chi \approx U$ so $W = 0$. Since either U or W is 0, I_χ is irreducible.

Furthermore, by the boxed formula,

$$\begin{aligned}\mathrm{Hom}_G(I_\chi, I_{\chi^w}) &= \mathrm{Hom}_G(\mathrm{Ind}_P^G(\chi), \mathrm{Ind}_P^G(\chi^w)) \\ &= \mathrm{Hom}_P(\chi, \chi^w) \;\oplus\; \mathrm{Hom}_{w^{-1}Pw\cap P}(\chi^w, \mathrm{Res}^P_{w^{-1}Pw\cap P}(\chi^w)) \\ &= 0 \oplus \mathbb{C},\end{aligned}$$

where the last equality follows from χ being regular and Schur's lemma. A corollary to Schur's Lemma [C.1] ensures that I_χ is irreducible if and only if $\dim_{\mathbb{C}}(\mathrm{End}(I_\chi)) = 1$, so I_χ is irreducible here. Furthermore, the calculation above shows that $I_\chi \approx I_{\chi^w}$.

Now, let $G = \mathrm{GL}(2, k)$ and χ be irregular. From Chapter 1, χ is an irregular M-character if and only if $\chi = \chi_0 \circ \det$ for some character χ_0 of $k^\times$. Since $\det : G \longrightarrow k^\times$ and $\chi_0 : k^\times \longrightarrow \mathbb{C}$, χ is a G-character and $\chi \in \{f : G \longrightarrow \mathbb{C} : f(pg) = p \cdot f(g)\} = I_\chi$. Thus χ spans a (one-dimensional) subrepresentation of I_χ, so $I_\chi = \chi \oplus W$ for G-representation $W = \chi^\perp$. Since χ is irregular, by Corollary 3.2, $\mathrm{End}_G(I_\chi) = \mathbb{C} \oplus \mathbb{C}$. Since the G-homomorphisms from χ to χ are $\mathbb{C}$ and the G-homomorphisms from W to W are *at least* $\mathbb{C}$, it follows that $\mathrm{End}_G(W) = \mathbb{C}$. Thus, $\dim_{\mathbb{C}}(\mathrm{End}_G(W)) = 1$, so by [C.1], W is irreducible. Thus, I_χ decomposes as χ and W for some irreducible G-representation W, as desired. □

3.4 Counting Results

Let $G = \mathrm{GL}(2, k)$. Corollary 3.3 has shown that for a character χ of M, the G-representation I_χ is irreducible (if χ is regular) or decomposes into irreducible parts (if χ is irregular). We will now provide a rough counting argument that will suggest that the irreducible G-representations described above, all related to I_χ, compose roughly half (!) of *all* the nonisomorphic G-representations. This argument, while more intuitive than rigorous, in fairly standard (see [4] for example).

However, we first present a theorem that is common to most literature on representation theory [4, 6, 10, 14].

Theorem 3.4. *Let H be a finite group. The number of (nonisomorphic) irreducible representations of H equals the number of conjugacy classes of H. Furthermore,*

$$\sum_V (\dim_{\mathbb{C}} V)^2 = |H|.$$

We can find the conjugacy classes of $\mathrm{GL}(2, k)$ by simply reasoning about diagonalizability and characteristic polynomials. Recall that $|k| = q$. The four (nonisomorphic) conjugacy classes are now described.

- Diagonal matrices of the form $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ for $x \neq y$ and $x, y \neq 0$, whose characteristic polynomials have two roots. Every distinct pair of eigenvalues $\{x, y\}$ yields a conjugacy class by standard linear algebra [3]. There are $q-1$ choices for nonzero x and y, so there are $\binom{q-1}{2}$ ways to pick x and y distinctly. Thus, there are $\frac{(q-1)(q-2)}{2}$ such conjugacy classes.

- Diagonal matrices of the form $\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$ for $x \neq 0$, whose characteristic polynomials each have a single root. Since there are $(q-1)$ options for nonzero x, there are $(q-1)$ such conjugacy classes.

- Matrices whose eigenvalues exist in k but are not diagonal. Note that this implies that the eigenvalues are equal; if they were not, the matrix would necessarily have two independent eigenvectors and conjugating by a matrix of these eigenvectors would diagonalize it. Any matrix of this form is conjugate to a matrix with the same eigenvalues that is non-diagonal. Thus, for a single conjugacy class, we can simply pick a representative matrix $\begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix}$ with $x \neq 0$. Since there are $(q-1)$ possible choices for x, there are $(q-1)$ such conjugacy classes.

- Matrices whose characteristic polynomials do not factor over k; that is, elements whose eigenvalues are in an extension field. There are $\frac{q(q-1)}{2}$ such conjugacy classes. For a discussion as to why, see [1] or [10].

By summing the classes above, there are q^2-1 total conjugacy classes of G. By Theorem 3.4, there are q^2-1 nonisomorphic irreducible representations of G.

In addition, we can count the number of irreducible G-representations that we have found through principal series. We will notice that the data is remarkably similar.

- If some character χ of M is regular, then I_χ is irreducible. Thus, we count the regular characters of M. Each pair of diagonal entries $\{x, y\}$ on some $m \in M$ corresponds to a separate character. There are $(q-1)$ options for $x \neq 0$ and $(q-2)$ options for $y \neq 0, x$, so there are $(q-1)(q-2)$ such characters. However, Corollary 3.3 also states that $I_\chi \approx I_{\chi^w}$. Since we are trying to find nonisomorphic I_χ by varying through χ, we have therefore double-counted. Thus, there are $\frac{(q-1)(q-2)}{2}$ total χ that yield nonisomorphic, irreducible I_χ.

- If a character χ of M is irregular, then we know that χ is a character of G (thus an irreducible G-representation) and $I_\chi = \chi \oplus W$ for irreducible G-representation W. Since $\chi = \chi_0 \circ \det$ for character χ_0 of $k^\times$, the number of distinct χ equals the number of distinct χ_0. Since the multiplicative group of a finite field is cyclic, $k^\times$ is cyclic of order $(q-1)$. Let g be the generator. Then $\chi_0(g)^{q-1} = 1_{\mathbb{C}}$, so $\chi_0(g)$ is any of the $(q-1)$ roots of unity. Thus, there are $(q-1)$ total χ_0, yielding $(q-1)$ irreducible G-representations χ.

- If χ is irregular, now consider the complement G-representation W. Since there are $(q-1)$ distinct χ, there are $(q-1)$ distinct complements W of χ.

Therefore, we can naturally match the conjugacy classes of G with irreducible representations of G, as summarized in the table below.

Representative	Number of Conj. Classes	Representation
$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$	$\frac{(q-1)(q-2)}{2}$	I_χ $\left(\frac{(q-1)(q-2)}{2} \text{ total}\right)$
$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$	$q-1$	χ $(q-1 \text{ total})$
$\begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix}$	$q-1$	W $(q-1 \text{ total})$
Remaining elements	$\frac{q(q-1)}{2}$	Remaining representations

Furthermore, we note that a conjugacy class of $\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$ has dimension 1, so pairing these $q-1$, one-dimensional classes with one-dimensional χ seems more suggestive than pairing them with W.

As shown later in [10], the remaining reperesentations (called *supercupsidal* representations) have at least order $q-1$. In [C.5] (Appendix C),we show that $|G| = (q^2-1)(q^2-q)$. Similarly, [C.6] shows that $|I_\chi| = q+1$. Thus, for when $I_\chi = \chi \oplus W$, $|W| = q$ since $|\chi| = 1$. In light of Theorem 3.4, we the compute that

$$|G| = \frac{(q-1)(q-2)}{2}(q+1)^2 + (q-1)1^2 + (q-1)q^2 + \frac{q(q-1)}{2}(q-1)^2$$

suggesting that the supercupsidal representations have order in fact equal to $q-1$.

It should be mentioned that Theorem 3.4 does not claim, in general, that the irreducible representations of G are isomorphic to the conjugacy classes of G; rather, it simply asserts that there are the same number of each. However, the matching in this example is so inevitable that we can safely say that the G-irreducibles of each type correspond to their matched conjugacy classes. The way in which the sum-of-squares formula (Theorem 3.4) fits together further confirms that we have matched correctly.

Therefore, given the irreducible G-representations I_χ, χ, W we found by examining principal series, we see that these representations compose

$$\frac{(q-1)(q-2)}{2} + (q-1) + (q-1) = \frac{q^2+q-2}{2}$$

representations of $\mathrm{GL}(2,k)$, which is roughly half of the q^2-1 total irreducibles of this group.

Appendix A

Frobenius Reciprocity for Finite Groups

Let G be a finite group and H be a subgroup of G. Let M be an H-representation and N be a G-representation. Recall that

$$\mathrm{Ind}_H^G(M) = \{f : G \longrightarrow M | f(hg) = h \cdot f(g)\} \text{ for all } h \in H, g \in G.$$

Also, $\mathrm{Ind}_H^G(M)$ is a G-representation with the following G-action,

$$(g \cdot f)(x) = f(xg) \text{ for all } x, g \in G.$$

We wish to show that

$$\mathrm{Hom}_H(\mathrm{Res}_H^G(N), M) \approx \mathrm{Hom}_G(N, \mathrm{Ind}_H^G(M)).$$

The proposed mapping and its inverse are

$$\begin{aligned} \varphi &\longmapsto \Phi_\varphi \quad \text{where } \Phi_\varphi(n)(g) = \varphi(g \cdot n), \\ \Phi &\longmapsto \varphi_\Phi \quad \text{where } \varphi_\Phi(n') = \Phi(n')(1_G). \end{aligned}$$

First, we check that these mappings are even sensible. Since Φ_φ needs to be an element of $\mathrm{Hom}_G(N, \mathrm{Ind}_H^G(M))$, we check that Φ_φ maps from N to $\mathrm{Ind}_H^G(M)$ such that:

- $\Phi_\varphi(n) : G \longrightarrow M$ is an H-map, and
- $\Phi_\varphi(g \cdot n) = g \cdot \Phi_\varphi(n)$.

By definition, $\Phi_\varphi(n)(g) = \varphi(g \cdot n) \in M$, so $\Phi_\varphi(n)$ maps from G to M, as desired. Also,

$$\begin{aligned} \Phi_\varphi(n)(hg) &= \varphi((hg) \cdot n) = \varphi(h \cdot (g \cdot n)) && \text{[def. of } \Phi_\varphi] \\ &= h \cdot \varphi(g \cdot n) && \text{[since } \varphi \text{ is an } H\text{-hom]} \\ &= h \cdot \Phi_\varphi(n)(g) && \text{[def. of } \Phi_\varphi,] \end{aligned}$$

so Φ_φ is an H-map. In addition, for any $g, x \in G$,

$$\Phi_\varphi(g \cdot n)(x) = \varphi(x \cdot (g \cdot n)) = \varphi((xg) \cdot n) = \Phi_\varphi(n)(xg) = (g \cdot \Phi_\varphi(n))(x),$$

where the last equality follows from the G-action on $\mathrm{Ind}_H^G(M)$. Since $x \in G$ was arbitrary, $\Phi_\varphi(g \cdot n) = g \cdot \Phi_\varphi(n)$, completing the bullet list.

Similarly, we need to check that $\varphi_\Phi \in \mathrm{Hom}_H(\mathrm{Res}_H^G(N), M)$. Note that, for $\Phi \in \mathrm{Hom}_G(N, \mathrm{Ind}_H^G(M))$, $\Phi(n)(1_G) \in M$. Since $\varphi_\Phi(n) = \Phi(n)(1_G)$, we have φ_Φ maps from N to M. Then φ_Φ maps $\mathrm{Res}_H^G(N)$ to M by simply restricting the allowed action on N. Finally, we just need to check that φ_Φ is an H-hom. Compute that

$$\begin{aligned}
\varphi_\Phi(h \cdot n) &= \Phi(h \cdot n)(1_G) && [\text{def. of } \varphi_\Phi] \\
&= (h \cdot \Phi(n))(1_G) && [\text{since } h \in G \text{ and } \Phi \text{ is a } G\text{-hom.}] \\
&= \Phi(n)(1_G h) = \Phi(n)(h 1_G) && [\text{def. of } H\text{-action on } \Phi] \\
&= h \cdot \Phi(n)(1_G) && [\text{since } \Phi(n) \text{ is an } H\text{-map}] \\
&= h \cdot \varphi_\Phi(n).
\end{aligned}$$

So, the proposed mappings Φ_φ and φ_Φ are sensible. By letting $\varphi \overset{a}{\longmapsto} \Phi_\varphi$ and $\Phi \overset{b}{\longmapsto} \varphi_\Phi$, observe that for all $n \in N, g \in G$,

$$\begin{aligned}
(a \circ b)(\Phi(n)(g)) &= \Phi_{\varphi_\Phi}(n)(g) && [\text{def. of } a,b] \\
&= \varphi_\Phi(g \cdot n) = \Phi(g \cdot n)(1_G) && [\text{def. of } \varphi_\Phi, \Phi_\varphi] \\
&= (g \cdot \Phi(n))(1_G) && [\text{since } \Phi \text{ is a } G\text{-hom}] \\
&= \Phi(n)(1_G g) && [\text{def. of } G\text{-action on } \Phi(n)] \\
&= \Phi(n)(g).
\end{aligned}$$

Thus, $a \circ b$ is the identity on $\mathrm{Hom}_G(N, \mathrm{Ind}_H^G(M))$. Similarly, for $n \in \mathrm{Res}_H^G(N)$,

$$\begin{aligned}
(b \circ a)(\varphi(n)) &= \varphi_{\Phi_\varphi}(n) && [\text{def. of } a, b] \\
&= \Phi_\varphi(n)(1_G) = \varphi(1_G \cdot n) && [\text{def. of } \varphi_\Phi, \Phi_\varphi] \\
&= \varphi(n),
\end{aligned}$$

so $b \circ a$ is the identity on $\mathrm{Hom}_H(\mathrm{Res}_H^G(N), M)$. Hence, a and b are inverses, so we have the desired bijection.

Appendix B

Constructions of Isotypes and Co-Isotypes

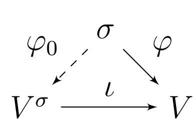
A pedagogical theme of this thesis is that mathematical objects are understood more clearly when characterized rather than constructed; that is, the eventual formula (construction) of an object can be made intuitive and motivated after we determine how that object *must* interact with its environment (characterization). For this reason, the characterizing definitions are always live.

However, constructions confirm that entities with the desired characterizing properties do in fact exist. For this reason, we check that the constructions of V^σ and V_σ as advertised do fulfill their respective characterizing properties. After doing so, we can safely refer to only the characteristic definitions of each.

Let H be a finite group. Let V be a finite-dimensional representation of H, and let σ be an irreducible H-representation. Given our category, *every arrow in this section is an H-map.*

B.1 Constructing the Isotype

Recall that the σ-isotype V^σ is *characterized* as the smallest subrepresentation of V with the following mapping property: for every $\varphi : \sigma \longrightarrow V$, there exists a unique $\varphi_0 : \sigma \longrightarrow V^\sigma$ such that $\varphi = \iota \circ \varphi_0$, where ι is simply an inclusion mapping. That is, we have the following commutative diagram.



We prove the existence of V^σ by *constructing* it as

$$V^\sigma = \sum_{\varphi \in \mathrm{Hom}_H(\sigma, V)} \mathrm{Im}(\varphi).$$

Let $\varphi : \sigma \longrightarrow V$ be given. In order to satisfy the mapping property, we need a $\varphi_0 : \sigma \longrightarrow V^\sigma$ such that for every $s \in \sigma$,

$$\varphi(s) = \iota(\varphi_0(s)) = \varphi_0(s).$$

However, this then *forces* φ_0 to be the map that takes s to $\varphi(s)$, so (given this only option) φ_0 is unique. Thus, the mapping property of V^σ is satisfied, so we only need to check that V^σ is the smallest subrepresentation of V with the mapping property.

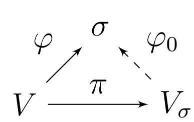
Let W be some subrepresentation of V that satisfies the mapping property. Suppose for contradiction that $V^\sigma = \sum_{\varphi \in \mathrm{Hom}_H(\sigma,V)} \mathrm{Im}(\varphi) \not\subseteq W$. Then in particular, there is some $\varphi : \sigma \longrightarrow V$ such that $\mathrm{Im}(\varphi) \not\subseteq W$. However, using this φ in the mapping property of W yields that there exists some unique $\varphi_0 : \sigma \longrightarrow W$ such that $\varphi = \iota \circ \varphi_0$. Let $y \in \mathrm{Im}(\varphi)$ so there exists some $x \in \sigma$ such that $\varphi(x) = y$. Then

$$y = \varphi(x) = \iota(\varphi_0(x)) = \varphi_0(x)$$

so $y \in \mathrm{Im}(\varphi_0) \subseteq W$. Since y was arbitrary, $\mathrm{Im}(\varphi) \subseteq W$, yielding a contradiction. Thus, for any W satisfying the mapping property, $V^\sigma \subseteq W$. This containment precisely means that V^σ is smaller than any W distinct from it. Therefore, V^σ is the smallest subrepresentation of V with the characteristic mapping property.

B.2 Constructing the Co-Isotype

Similarly, the σ-co-isotype V_σ is characterized as the smallest quotient representation of V with the following mapping property: for every $\varphi : V \longrightarrow \sigma$, there exists a unique $\varphi_0 : V_\sigma \longrightarrow \sigma$ such that $\varphi = \varphi_0 \circ \pi$, where $\pi : V \longrightarrow V_\sigma$ is a projection mapping. Thus, we have the following commutative diagram.



Here, we prove the existence of the σ-co-isotype by *constructing* it as

$$V_\sigma = V/K, \quad K = \bigcap_{\varphi \in \mathrm{Hom}_H(V,\sigma)} \ker(\varphi).$$

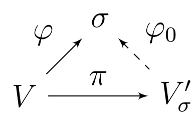
First, we show that V_σ satisfies the desired mapping property. Let $\varphi : V \longrightarrow \sigma$ be given. In order to satisfy the mapping property, we need a $\varphi_0 : V_\sigma \longrightarrow \sigma$ such that for all $v \in V$,

$$\varphi(v) = \varphi_0(\pi(v)) = \varphi_0(v + K),$$

since π is a quotient map. Hence, $\varphi_0 : V/K \longrightarrow \sigma$ is *forced* to be $\varphi_0(v + K) = \varphi(v)$, showing that φ_0 is unique. To show its existence, we simply check that φ_0 is well-defined. Suppose $v_1 + K = v_2 + K$. Then $(v_1 - v_2) \in K$, so by definition of K,

$\varphi(v_1 - v_2) = 0$. Since φ is linear, $\varphi(v_1) = \varphi(v_2)$. Thus, $\varphi_0(v_1 + K) = \varphi_0(v_2 + K)$, as desired. Given this unique, well-defined φ_0, the mapping property of V_σ is satisfied.

We now confirm that V_σ is constructed is the *smallest* quotient representation of V with the characteristic mapping property. Suppose there exists some other V'_σ that satisfies the mapping property. Then, we have the following diagram.



Let $N = \ker(\pi)$, so by the first isomorphism Theorem, $V/N = V'_\sigma$. Then $\varphi : V \longrightarrow \sigma$ *factors through* V/N, so by Section 1.8, $N \subseteq \ker(\varphi)$. Since φ was arbitrary, $N \subseteq K$. Accordingly, $V'_\sigma = V/N \twoheadrightarrow V/K = V_\sigma$. This surjection precisely means that V_σ is smaller than some arbitrary V'_σ with the mapping property, meaning that V_σ as constructed is the smallest such quotient representation.

Appendix C

Appendix of Small Proofs

Lemma C.1. Let V be a complex, finite-dimensional representation of finite group G. Then V is irreducible if and only if $\dim_{\mathbb{C}}(\mathrm{End}_G(V)) = 1$.

Proof. ($\Rightarrow$) If V is irreducible, $\mathrm{End}_G(V) = \mathbb{C}$ by Schur's Lemma.

($\Leftarrow$) By the contrapositive, if V is reducible, then V has a proper, nontrivial G-subrepresentation U. So the projection $(\pi : V \longrightarrow U) \in \mathrm{End}_G(V)$ is nonscaling (for otherwise, $V \approx U$ or $U = 0$), so $1 < \dim_{\mathbb{C}}(\mathrm{End}_G(V))$. □

Lemma C.2. Recall that for finite-dimensional G-representation V, π_{V^*} is

$$V^* \longrightarrow (V^*)_N, \quad v \longmapsto \int_N n \cdot v \; dn + S_{V^*}$$

for the unipotent radical N and N-fixed $S_{V^*} = \mathrm{span}\{n \cdot v - v : v \in V^*, n \in N\}$. Let $\mu \in V^*$ be N-fixed. Then μ is in the kernel of π_{V^*} if and only if $\int_N n \cdot \mu \; dn = 0$.

Proof. Observe that

$$\begin{aligned} \int_N n \cdot \mu \; dn = 0 &\Leftrightarrow \mu - \frac{1}{|N|} \int_N n \cdot \mu \; dn = \mu \\ &\Leftrightarrow \frac{\mu}{|N|} \int_N dn - \frac{1}{|N|} \int_N n \cdot \mu \; dn = \mu \quad \left[\text{since } \frac{1}{|N|} \int_N dn = 1\right] \\ &\Leftrightarrow \frac{1}{|N|} \int_N (\mu - n \cdot \mu) \; dn = \mu. \end{aligned}$$

Since $\int_N (\mu - n \cdot \mu) \; dn \in S_{V^*}$[1] we have that $\mu \in S_{V^*}$. Hence,

$$\begin{aligned} \int_N n \cdot \mu \; dn = 0 &\Leftrightarrow \mu \in S_{V^*} \\ &\Leftrightarrow \int_N n \cdot \mu \; dn \in S_{V^*} && [\text{since } \mu \text{ is } N\text{-fixed}] \\ &\Leftrightarrow \pi_{V^*}(\mu) \in S_{V^*} && [\text{def. of } \pi_{V^*}] \\ &\Leftrightarrow \mu \in \ker(\pi_{V^*}), \end{aligned}$$

concluding the proof. □

[1] Recall that integrals over finite N are equivalent to finite sums over N.

Lemma C.3. Fix a coset AhB. Then the map

$$A\backslash AhB \longrightarrow (h^{-1}Ah \cap B)\backslash B, \quad Ahb \longmapsto (h^{-1}Ah \cap B)b.$$

is a bijection.

Proof. Clearly surjectivity holds, so we check injectivity. Suppose $(h^{-1}Ah \cap B)b_1 = (h^{-1}Ah \cap B)b_2$. Then $b_1 b_2^{-1} \in h^{-1}Ah \cap B$, so $b_1 b_2^{-1} \in h^{-1}Ah$. Thus, $hb_1(hb_2)^{-1} \in A$, so $Ahb_1 = Ahb_2$, as desired. □

Lemma C.4. Let X, Y be finite-dimensional vector spaces, and let B be a finite group. Then,

$$\mathrm{Hom}_B(X, Y) \approx \mathrm{Hom}_B(Y^*, X^*).$$

Proof. We first construct a bijection between $\mathrm{Hom}_B(X, Y^*)$ and $\mathrm{Hom}_B(X \otimes Y, \mathbb{C})$. Let $\lambda \in \mathrm{Hom}_B(X, Y^*)$ so that $\lambda(x) \in Y^*$. Then, we have the map

$$\begin{aligned} \mathrm{Hom}_B(X, Y^*) &\xrightarrow{\gamma} \mathrm{Hom}_B(X \otimes Y, \mathbb{C}) \\ \lambda &\longmapsto \big(x \otimes y \longmapsto (\lambda(x))(y)\big) \end{aligned}$$

which is a homomorphism. Similarly, there is the map

$$\begin{aligned} \mathrm{Hom}_B(X \otimes Y, \mathbb{C}) &\xrightarrow{\zeta} \mathrm{Hom}_B(X, Y^*) \\ \mu &\longmapsto \big(x \longmapsto (y \longmapsto \mu(x \otimes y)\big). \end{aligned}$$

To confirm that $\mathrm{Hom}_B(X, Y^*) \approx \mathrm{Hom}_B(X \otimes Y, \mathbb{C})$, we need to check that $\gamma \circ \zeta = \mathrm{id}_{\mathrm{Hom}_B(X \otimes Y, \mathbb{C})}$ and $\zeta \circ \gamma = \mathrm{id}_{\mathrm{Hom}_B(X, Y^*)}$. Observe that

$$\gamma(\zeta(\mu)) = \gamma\big(x \longmapsto (y \longmapsto \mu(x \otimes y)\big) = (x \otimes y \longmapsto \mu(x \otimes y)) = \mu.$$

Similarly,

$$\zeta(\gamma(\lambda)) = \zeta\big(x \otimes y \longmapsto (\lambda(x))(y)\big) = \big(x \longmapsto (y \longmapsto \lambda(x)(y))\big) = \lambda.$$

Thus, $\mathrm{Hom}_B(X, Y^*) \approx \mathrm{Hom}_B(X \otimes Y, \mathbb{C})$. By similar calculations, we can show that $\mathrm{Hom}_B(Y, X^*) \approx \mathrm{Hom}_B(X \otimes Y, \mathbb{C})$. So,

$$\mathrm{Hom}_B(X, Y^*) \approx \mathrm{Hom}_B(Y, X^*).$$

Since Y is finite-dimensional, $Y^{**} = Y$. Thus, we have that

$$\mathrm{Hom}_B(X, Y) \approx \mathrm{Hom}_B(X, Y^{**}) \approx \mathrm{Hom}_B(Y^*, X^*).$$

□

Lemma C.5. Let $G = \mathrm{GL}(2, k)$ for field k with $|k| = q$. Then

$$|\mathrm{GL}(2, k)| = (q^2 - 1)(q^2 - q).$$

Proof. Note that for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, there are $q^2 - 1$ options for row (a, b), since this row needs to be nonzero for $\det(g) \neq 0$. Similarly, there are $q^2 - q$ options for (c, d) since for $\det(g) \neq 0$, we need (c, d) to *not* be one of the q multiples of (a, b) (including the zero multiple). Thus, $|\mathrm{GL}(2, k)| = (q^2 - 1)(q^2 - q)$. □

Lemma C.6. Regardless of whether G is $\mathrm{GL}(2, k)$ or $\mathrm{SL}(2, k)$, the complex dimension of I_χ is

$$\dim_{\mathbb{C}}(I_\chi) = |P \backslash G| = q + 1$$

where q is the order of k.

Proof. Clearly $\dim_{\mathbb{C}}(I_\chi) = |P \backslash G|$. By [C.5], $|\mathrm{GL}(2, k)| = (q^2 - 1)(q^2 - q)$. Let $p = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in P$. Then, since $p \in \mathrm{GL}(2, k)$, we need $d \neq 0$ so that $\det(p) \neq 0$. Given that $d \neq 0$, there is no restriction on b. Hence, there are q options for b and $q - 1$ options for d, yielding $q(q - 1)$ options for column (b, d). Then column $(a, 0)$ is guaranteed to not be a (nonzero) multiple of (b, d) since $d \neq 0$. Thus, the only restriction is that $a \neq 0$ so that $(a, 0)$ is not a zero-column. Thus, there are $(q - 1)$ options for a, meaning $|P| = q(q - 1)(q - 1) = q(q - 1)^2$. So, $|P \backslash G| = \frac{(q^2-1)(q^2-q)}{(q-1)^2 q} = q + 1$. When $G = \mathrm{SL}(2, k)$, the counting argument is similar. □

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