Incongruent Restricted Disjoint Covering Systems

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Introduction

A *cover* is a collection of finitely many linear congruences $\{a_1 (\text{mod } n_1), ..., a_k (\text{mod } n_k)\}$ such that every integer is contained in at least one linear congruence. We now examine specific types of covers:

1. A cover is *incongruent* if all of its moduli are distinct.
2. A cover is *disjoint* if every integer is contained in one unique linear congruence.
3. A cover is *restricted* if every integer on a finite interval $[1, n]$ is contained in at least one of its linear congruences.

The Mirsky-Newman Theorem [5] states that it is impossible for a covering of the integers to be both incongruent and disjoint; however, systems of congruences that cover only finite intervals can meet both of these requirements simultaneously. These special systems are known as incongruent restricted disjoint covering systems (IRDCS).

IRDCS have the additional requirement that each congruence must cover at least two integers within the given interval.

Background Information

Definition 1: The *length* of an IRDCS is the number of integers that it covers; an IRDCS on $[1, n]$ has length n.

Definition 2: The *order* of an IRDCS is the number of congruence classes it contains.

Definition 3: Given an IRDCS A of length n, the *sequential notation* for A will consist of a sequence of n integers. In this sequence, the i^{th} term will equal the modulus of the congruence class which covers i.

Example 4: A comparison between traditional congruence notation and sequential notation

$$\begin{array}{l} n \equiv 1 \pmod 6 \\ n \equiv 2 \pmod 9 \\ n \equiv 0 \pmod 3 \\ n \equiv 0 \pmod 4 \\ n \equiv 0 \pmod 5 \end{array} \Leftrightarrow \{6, 9, 3, 4, 5, 3, 6, 4, 3, 5, 9\}$$

Definition 5: A *family* is a collection of IRDCS such that all members satisfy an additional requirement beyond that of the generalized IRDCS definition.

Definition 6: Let $A = \{S(m_i, a_i) : i = 1, ..., k\}$ be an IRDCS of length n. Then the reversal of A, denoted by A', is $A' = \{S(m_i, n + 1 - a_i) : i = 1, ..., k\}$. The sequential notation of a reversal will be exactly the reverse of the sequential notation of the original IRDCS.

Theorem 7 (G. Myerson, J. Poon, J. Simpson. 2009): No IRDCS equals its reversal.

Exploring IRDCS and the 9-6-3 Construction

Theorem 8: Let p be a prime, $a \in \mathbb{Z}$ such that $p \nmid a$. Then for an IRDCS with a and p as moduli, the congruence class corresponding to a can cover at most $p - 1$ integers without clashing with an integer covered by the congruence class corresponding to p.

Definition 9: A *9-6-3 construction* is an IRDCS of length 18 or greater with the sequence of moduli *9, 6, 3* somewhere in its sequential notation. [1]

If we have the sequence *9, 6, 3* appearing anywhere in the sequential notation of an IRDCS of length 18 or greater, the following pattern will occur:

$$...,9,6,3,\_,\_,3,\_,6,3,9,\_,3,\_,6,3,\_,\_,3...$$

Now, we sharpen the lower bound on the possible lengths of 9-6-3 constructions in [1].

Theorem 10: There exists no IRDCS of length 18 that adheres to the 9-6-3 construction.

Theorem 11: Any IRDCS with 3, 6, and 9 as moduli must be either a 9-6-3 IRDCS or its reversal.

Consider an IRDCS in sequential notation of length $n > 18$. When using moduli 3, 6, 9, without loss of generality, observe a possible portion of the sequence beginning with modulus 3. Note the following choices for the placements of the moduli 6 and 9:

$$\begin{array}{l} \ldots 3 - - 3 - - 3 - - 3 - - 3 - - 3 - - \ldots \\ \ldots 3 - 6\ 3 - - 3 - 6\ 3 - - 3 - 6\ 3 - - \ldots \\ \ldots 3\ 9\ 6\ 3 - - 3 - 6\ 3\ 9 - 3 - 6\ 3 - - \ldots \end{array}$$

or

$$\begin{array}{l} \ldots 3 - - 3 - - 3 - - 3 - - 3 - - 3 - - \ldots \\ \ldots 3 - 6\ 3 - - 3 - 6\ 3 - - 3 - 6\ 3 - - \ldots \\ \ldots 3 - 6\ 3\ 9 - 3 - 6\ 3 - - 3\ 9\ 6\ 3 - - \ldots \end{array}$$

or

$$\begin{array}{l} \ldots 3 - - 3 - - 3 - - 3 - - 3 - - 3 - - \ldots \\ \ldots 3 - 6\ 3 - - 3 - 6\ 3 - - 3 - 6\ 3 - - \ldots \\ \ldots 3 - 6\ 3\ 9 - 3 - 6\ 3\ 9 - 3\ 9\ 6\ 3 - - \ldots \end{array}$$

Notice that all of these possibilities are 9-6-3 constructions. Similarly, the other choice of residue modulo 6 yields the reversal of a 9-6-3 construction:

$$\begin{array}{l} \ldots 3 - - 3 - - 3 - - 3 - - 3 - - 3 - - \ldots \\ \ldots 3\ 6 - 3 - - 3\ 6 - 3 - - 3\ 6 - 3 - - \ldots \\ \ldots 3\ 6\ 9\ 3 - - 3\ 6 - 3 - 9\ 3\ 6 - 3 - - \ldots \end{array}$$

or

$$\begin{array}{l} \ldots 3 - - 3 - - 3 - - 3 - - 3 - - 3 - - \ldots \\ \ldots 3\ 6 - 3 - - 3\ 6 - 3 - - 3\ 6 - 3 - - \ldots \\ \ldots 3\ 6 - 3\ 9 - 3\ 6 - 3 - - 3\ 6\ 9\ 3 - - \ldots \end{array}$$

or

$$\begin{array}{l} \ldots 3 - - 3 - - 3 - - 3 - - 3 - - 3 - - \ldots \\ \ldots 3\ 6 - 3 - - 3\ 6 - 3 - - 3\ 6 - 3 - - \ldots \\ \ldots 3\ 6 - 3 - - 3\ 6\ 9\ 3 - - 3\ 6 - 3 - 9 \ldots \end{array}$$

Open Question (Emanuel, 2011): Does there exist some $N \in \mathbb{N}$ such that there exists a 9-6-3 construction for all lengths $n \geq N$? [1]

We have found 9-6-3 constructions for all lengths from 19-23 and 25-482. For selected lengths, we present the number of 9-6-3 constructions and their associated orders.

Selected Results for 9-6-3 IRDCS

Length (n)	Number of 9-6-3 IRDCS	Order
18	0	-
19	4	7
20	3	7
21	1	7
22	1	8
23	1	8
24	0	-
25	6	8
26	7	8
27	5	8
28	8	8,9
29	17	9
30	19	9
31	16	9
32	70	9,10
33	103	9,10
34	75	9,10

Table 1: Number of 963 IRDCS of various lengths and their respective orders.

We note that Theorem 11 implies that the existence of an IRDCS of length n using the moduli 3, 6, and 9 will guarantee the existence of a 9-6-3 construction with length n.

For IRDCS of longer lengths, it quickly becomes infeasible to compute 9-6-3 constructions manually. Therefore, we (with E. Wesson) create a program to compute all IRDCS of a given length based on the algorithm detailed in [1] and [2]. The algorithm builds an IRDCS through a process called *backtracking*.

We modify the program by altering the print statement to output only IRDCS that use the moduli 3, 6, and 9. Additionally, we can exclude moduli that are relatively prime to 3 and cover at least 3 integers in a given IRDCS.

References

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