STATS 310 HOMEWORK 1

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Question 1.1.4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $A, B, A_i \in \mathcal{F}$ be events. Then the following statements hold:

(a) (Monotonicity) If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
(b) (Countable sub-additivity) If $A \subseteq \cup_{i=1}^{\infty} A_i$ then $P(A) \leq \sum_{i=1}^{\infty} P(A_i)$.
(c) (Continuity from below) If $A_i \uparrow A$ (that is, $A_1 \subseteq A_2 \subseteq \cdots$ and $A = \cup_{i=1}^{\infty} A_i$) then $\mathbb{P}(A_i) \uparrow \mathbb{P}(A)$.
(d) (Continuity from above) If $A_i \downarrow A$ (that is, $A_1 \supseteq A_2 \supseteq \cdots$ and $A = \cap_{i=1}^{\infty} A_i$) then $\mathbb{P}(A_i) \downarrow \mathbb{P}(A)$.

Proof of (a). Note that $B = (B \cap A) \cup (B \setminus A) = A \cup (B \setminus A)$ (since $A \subseteq B$). By finite additivity,

$$\begin{aligned} \mathbb{P}(B) &= \mathbb{P}(A \cup (B \setminus A)) = \mathbb{P}(A) + \mathbb{P}(B \setminus A) \\ &\geq \mathbb{P}(A) + 0 = \mathbb{P}(A) \end{aligned}$$

where the last inequality follows since $\mathbb{P}(B \setminus A) \geq 0$ by non-negativity of measure. Thus, $\mathbb{P}(A) \leq \mathbb{P}(B)$. $\square$

Proof of (b). Let $B_1 = A_1$ and $B_k = A_k \setminus (\cup_{i=1}^{k-1} A_i)$. Then the $\{B_k\}$ are disjoint. (Suppose there exists $y \in B_k \cap B_j$ for $k \neq j$. Without loss of generality, let $k < j$. Then $y \in A_k \setminus (\cup_{i=1}^{k-1} A_i) \subseteq A_k$. Also, $y \in A_j \setminus (\cup_{i=1}^{j-1} A_i) \subseteq A_j \setminus A_k$ since $k \in \{1, \dots, j-1\}$. So $y \in A_k \cap A_k^c$, which is a contradiction.) Since each $B_k \subseteq A_k$, we have $\cup_{k=1}^{\infty} B_k \subseteq \cup_{k=1}^{\infty} A_k$. Now suppose $x \in \cup_{k=1}^{\infty} A_k$, so $x \in A_j$ for some $j \in \mathbb{N}$. Then the set $\{j \in \mathbb{N} \mid x \in A_j\}$ is nonempty. Letting $\ell = \min\{j \in \mathbb{N} \mid x \in A_j\}$, we have $x \in A_\ell$ and $x \notin A_k$ for any $k < \ell$, meaning $x \in B_\ell = \left(A_\ell \setminus \cup_{i=1}^{\ell-1} A_i\right) \subseteq \cup_{k=1}^{\infty} B_k$. Therefore, $\cup_{k=1}^{\infty} A_k \subseteq \cup_{k=1}^{\infty} B_k$, and by mutual containment

$$\cup_{k=1}^{\infty} A_k = \cup_{k=1}^{\infty} B_k$$

for disjoint $\{B_k\}$. Then

$$\begin{aligned} \mathbb{P}(A) &\leq \mathbb{P}(\cup_{i=1}^{\infty} A_i) && [\text{by part (a) since } A \subseteq \cup_{i=1}^{\infty} A_i] \\ &= \mathbb{P}(\cup_{i=1}^{\infty} B_i) && [\text{since } \cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} B_i] \\ &= \sum_{i=1}^{\infty} \mathbb{P}(B_i) && [\text{by countable additivity}] \\ &\leq \sum_{i=1}^{\infty} \mathbb{P}(A_i) && [\text{by part (a) since each } B_i \subseteq A_i, \text{ so } \mathbb{P}(B_i) \leq \mathbb{P}(A_i)] \end{aligned}$$

as desired. $\square$

Proof of (c). Since $A_i \subseteq \cup_{i=1}^{\infty} A_i = A$, by part (a) we have $\mathbb{P}(A_i) \leq \mathbb{P}(A)$ for every $i \in \mathbb{N}$.

As before, let $B_1 = A_1$ and $B_k = A_k \setminus (\cup_{i=1}^{k-1} A_i)$, so the $\{B_k\}$ are disjoint and $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} B_i$. Furthermore, $\cup_{i=1}^{k} B_i = A_k$ (see Lemma 1.1.4 following this problem). Then

$$\begin{aligned}
\lim_{k\to\infty} \mathbb{P}(A_k) &= \lim_{k\to\infty} \mathbb{P}(\cup_{i=1}^{k} B_i) && [\text{since } \cup_{i=1}^{k} B_i = A_k] \\
&= \lim_{k\to\infty} \sum_{i=1}^{k} \mathbb{P}(B_i) && [\text{by finite additivity}] \\
&= \sum_{i=1}^{\infty} \mathbb{P}(B_i) = \mathbb{P}(\cup_{i=1}^{\infty} B_i) && [\text{by countable additivity}] \\
&= \mathbb{P}(\cup_{i=1}^{\infty} A_i) && [\text{since } \cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} A_i] \\
&= \mathbb{P}(A) && [\text{since } \cup_{i=1}^{\infty} A_i = A].
\end{aligned}$$

Since each $\mathbb{P}(A_i) \leq \mathbb{P}(A)$ and $\lim_{i\to\infty} \mathbb{P}(A_i) = \mathbb{P}(A)$, we conclude that $\mathbb{P}(A_i) \uparrow \mathbb{P}(A)$. $\square$

Proof of (d). Since each $A_{i+1} \subseteq A_i$, we have $A_i^c \subseteq A_{i+1}^c$.[1] Therefore

$$A_1^c \subseteq A_2^c \subseteq \cdots \subseteq A_i^c \subseteq \cdots$$

and $\cup_{i=1}^{\infty} A_i^c = (\cap_{i=1}^{\infty} A_i)^c = A^c$ since $A = \cap_{i=1}^{\infty} A_i$. Hence, $A_i^c \uparrow A^c$ and by part (c) $\mathbb{P}(A_i^c) \uparrow \mathbb{P}(A^c)$. By definition of $\uparrow$, each $\mathbb{P}(A_i^c) \leq \mathbb{P}(A^c)$ and $\lim_{i\to\infty} \mathbb{P}(A_i^c) = \mathbb{P}(A^c)$. From $\mathbb{P}(A_i^c) = 1 - \mathbb{P}(A_i)$ and $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$,[2] we have $1 - \mathbb{P}(A_i) \leq 1 - \mathbb{P}(A)$ and $\lim_{i\to\infty}(1 - \mathbb{P}(A_i)) = 1 - \mathbb{P}(A)$. By rearranging terms, $\mathbb{P}(A_i) \geq \mathbb{P}(A)$ for every $i \in \mathbb{N}$ and $\lim_{i\to\infty} \mathbb{P}(A_i) = \mathbb{P}(A)$. Therefore, $\mathbb{P}(A_i) \downarrow \mathbb{P}(A)$. $\square$

Lemma 1.1.4. For events $A, B, A_i \in \mathcal{F}$, suppose $A_i \uparrow A$. Let $B_1 = A_1$ and $B_k = A_k \setminus (\cup_{i=1}^{k-1} A_i)$. Then $\cup_{i=1}^{k} B_i = A_k$.

Proof. Since each $A_i \subseteq A_{i+1}$ (by definition of $\uparrow$), we have $\cup_{i=1}^{k-1} A_i = A_{k-1}$. Thus, $B_k = A_k \setminus (\cup_{i=1}^{k-1} A_i) = A_k \setminus A_{k-1}$. We proceed by induction over k. For $k = 1$, note that $\cup_{i=1}^{k} B_i = B_1 = A_1$, completing the base case.

For the inductive step, assume $\cup_{i=1}^{k-1} B_i = A_{k-1}$. Then

$$\begin{aligned}
\cup_{i=1}^{k} B_i &= \cup_{i=1}^{k-1} B_i \cup B_k \\
&= A_{k-1} \cup B_k && [\text{by inductive hypothesis}] \\
&= A_{k-1} \cup (A_k \setminus A_{k-1}) && [\text{since } B_k = A_k \setminus A_{k-1}] \\
&= (A_k \cap A_{k-1}) \cup (A_k \setminus A_{k-1}) && [\text{since } A_{k-1} \subseteq A_k] \\
&= A_k
\end{aligned}$$

completing the inductive step. Thus, $\cup_{i=1}^{k} B_i = A_k$. $\square$

[1] This follows since in *general* $A \subseteq B \Leftrightarrow (x \in A \Rightarrow x \in B) \Leftrightarrow (x \notin B \Rightarrow x \notin A)$ by the contrapositive $\Leftrightarrow (x \in B^c \Rightarrow x \in A^c) \Leftrightarrow B^c \subseteq A^c$.

[2] For any $A \in 2^{\Omega}$ in a probability space, $\mathbb{P}(A) + \mathbb{P}(A^c) = \mathbb{P}(A \sqcup A^c) = \mathbb{P}(\Omega) = 1$ by finite additivity. Thus $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

Question 1.1.13. (a) Let $\{\mathcal{A}\}_{\alpha\in\Gamma}$ be a collection of σ-algebras, where the index set Γ is not necessarily countable. Then $\cap_{\alpha\in\Gamma}A_\alpha$ is a σ-algebra as well.

Proof. Since each A_α is a σ-algebra, we have $\emptyset, \Omega \in A_\alpha$ for all $\alpha \in \Gamma$. Thus, $\emptyset, \Omega \in \cap_{\alpha\in\Gamma}A_\alpha$. Now suppose $A \in \cap_{\alpha\in\Gamma}A_\alpha$. Then $A \in A_\alpha$ for every α, and by closure under complements $A^c \in A_\alpha$ for all α; hence, $A^c \in \cap_{\alpha\in\Gamma}A_\alpha$. Finally, suppose $B_i \in \cap_{\alpha\in\Gamma}A_\alpha$ for $i \in \mathbb{N}$. For a given α, $B_i \in A_\alpha$ for all $i \in \mathbb{N}$, so $\cup_{i=1}^\infty B_i \in A_\alpha$ by closure under countable union; however, since this holds for any $\alpha \in \Gamma$, $\cup_{i=1}^\infty B_i \in \cap_{\alpha\in\Gamma}A_\alpha$. Therefore, $\cap_{\alpha\in\Gamma}A_\alpha$ is a σ-algebra. □

(b) Let $\mathcal{H}, \mathcal{G}$ be σ-algebras with $\mathcal{H} \subseteq \mathcal{G}$. For $H \in \mathcal{H}$, define

$$\mathcal{H}^H = \{A \in \mathcal{G} \mid A \cap H \in \mathcal{H}\}.$$

Then $\mathcal{H}^H$ is a σ-algebra for any $H \in \mathcal{H}$.

Proof. Note that $\emptyset, \Omega \in \mathcal{G}$ (by definition of a σ-algebra), and for any $H \in \mathcal{H}$ we have $\emptyset \cap H = \emptyset \in \mathcal{H}$ and $\Omega \cap H = H \in \mathcal{H}$. Thus, $\emptyset, \Omega \in \mathcal{H}^H$. Now suppose $A \in \mathcal{H}^H$, so $A \in \mathcal{G}$ and $A \cap H \in \mathcal{H}$. Note that $A^c \in \mathcal{G}$ since $\mathcal{G}$ must be closed under complements. Observe that $A^c \cap H = (A \cap H)^c \cap H$, because

$$(A \cap H)^c \cap H = (A^c \cup H^c) \cap H = (A^c \cap H) \cup (H^c \cap H) = (A^c \cap H) \cup \emptyset = A^c \cap H.$$

Since $(A \cap H), H \in \mathcal{H}$, we have $A^c \cap H \in \mathcal{H}$ by closure under complements and finite intersection. Thus, $A^c \in \mathcal{H}^H$. Finally, let $A_i \in \mathcal{H}^H$ for $i \in \mathbb{N}$. Then each $A_i \in \mathcal{G}$ and $A_i \cap H \in \mathcal{H}$. Note that $\cup_{i=1}^\infty A_i \in \mathcal{G}$ by closure under countable union. Furthermore,

$$(\cup_{i=1}^\infty A_i) \cap H = \cup_{i=1}^\infty (A_i \cap H) \in \mathcal{H},$$

again by closure under countable union. Hence, $\cup_{i=1}^\infty A_i \in \mathcal{H}^H$. We conclude that $\mathcal{H}^H$ is a σ-algebra. □

(c) Consider the map f on $\mathcal{H}$ sending $H \overset{f}{\mapsto} \mathcal{H}^H$. Then f is non-increasing with respect to set-inclusions; that is, $A \subset B \Rightarrow f(A) \supseteq f(B)$ for all $A, B \in \mathcal{H}$. Furthermore, $\mathcal{H}^{H\cup H'} = \mathcal{H}^H \cap \mathcal{H}^{H'}$ for all $H, H' \in \mathcal{H}$.

Proof. Let $H \subseteq H'$ for $H, H' \in \mathcal{H} \subseteq \mathcal{G}$. Let $A \in f(H') = \mathcal{H}^{H'}$, so $A \in \mathcal{G}$ and $A \cap H' \in \mathcal{H}$. Since $H \subseteq H'$, $H = H' \cap H$ and so $A \cap H = A \cap (H' \cap H) = (A \cap H') \cap H$. Since $H, (A \cap H') \in \mathcal{H}$, by closure under intersection, $A \cap H \in \mathcal{H}$ as well. Thus, $A \in \mathcal{H}^H$ for arbitrary $A \in \mathcal{H}^{H'}$ and so $f(H') = \mathcal{H}^{H'} \subseteq \mathcal{H}^H = f(H)$. Therefore, f is non-increasing with respect to set inclusions.

Now let $H, H' \in \mathcal{H}$ be arbitrary. We will show that $\mathcal{H}^{H\cup H'} = \mathcal{H}^H \cap \mathcal{H}^{H'}$ by mutual containment. Let $A \in \mathcal{H}^H \cap \mathcal{H}^{H'}$. Then $A \in \mathcal{G}$ and $(A \cap H), (A \cap H') \in \mathcal{H}$. So

$$A \cap (H \cup H') = (A \cap H) \cup (A \cap H') \in \mathcal{H}$$

by closure under finite intersection. Hence, $A \in \mathcal{H}^{H\cup H'}$ for arbitrary $A \in \mathcal{H}^H \cap \mathcal{H}^{H'}$ and so $\mathcal{H}^H \cap \mathcal{H}^{H'} \subseteq \mathcal{H}^{H\cup H'}$.

Furthermore, since $H, H' \subseteq H \cup H'$ and f is non-increasing, $\mathcal{H}^{H\cup H'} \subseteq \mathcal{H}^H$ and $\mathcal{H}^{H\cup H'} \subseteq \mathcal{H}^{H'}$. Hence, $\mathcal{H}^{H\cup H'} \subseteq \mathcal{H}^H \cap \mathcal{H}^{H'}$. By mutual containment, $\mathcal{H}^{H\cup H'} = \mathcal{H}^H \cap \mathcal{H}^{H'}$. □

Question 1.1.21. For any $d < \infty$,

$$\mathcal{B}_{\mathbb{R}^d} = \mathcal{B}_\mathbb{R} \times \cdots \times \mathcal{B}_\mathbb{R} = \sigma(\{(a_1, b_1) \times \cdots \times (a_d, b_d) \mid a_i, b_i \in \mathbb{R}, a_i < b_i\})$$

Proof. By definition, $\mathcal{B}_{\mathbb{R}} \times \cdots \times \mathcal{B}_{\mathbb{R}} = \sigma(\{A_1 \times \cdots \times A_d \mid A_i \in \mathcal{B}_{\mathbb{R}}\})$. Let

$$\Sigma_{\text{prod}} = \sigma(\{(a_1, b_1) \times \cdots \times (a_d, b_d) \mid a_i, b_i \in \mathbb{R}, a_i < b_i\}).$$

We first show that $\mathcal{B}_{\mathbb{R}} \times \cdots \times \mathcal{B}_{\mathbb{R}} = \Sigma_{\text{prod}}$ by mutual containment. Let $A_1 \times \cdots \times A_d$ (with each $A_i \in \mathcal{B}_{\mathbb{R}}$) be arbitrary but fixed. Similarly, fix intervals $(a_2, b_2), \ldots, (a_d, b_d) \subset \mathbb{R}$ with $a_i < b_i$. Consider

$$\mathcal{C}_1 = \{A \in \mathcal{B}_{\mathbb{R}} \mid A \times (a_2, b_2) \times \cdots \times (a_d, b_d) \in \Sigma_{\text{prod}}\}.$$

We now show that $\mathcal{C}_1$ is a σ-algebra. Since $(-n, n) \times (a_2, b_2) \times \cdots \times (a_d, b_d) \in \Sigma_{\text{prod}}$ for $n \in \mathbb{N}$, by closure under countable union,

$$\cup_{n=1}^{\infty} [(-n, n) \times (a_2, b_2) \times \cdots \times (a_d, b_d)] = [\cup_{n=1}^{\infty} (-n, n)] \times (a_2, b_2) \times \cdots \times (a_d, b_d)$$

is also an element of Σ_{prod}. However, $\cup_{n=1}^{\infty} (-n, n) = \mathbb{R}$ which means that $\mathbb{R} \times (a_2, b_2) \times \cdots \times (a_d, b_d) \in \Sigma_{\text{prod}}$ and thus $\mathbb{R} = \Omega \in \mathcal{C}_1$. Similarly,

$$\emptyset \times (a_2, b_2) \times \cdots \times (a_d, b_d) = \emptyset \in \Sigma_{\text{prod}},$$

so $\emptyset \in \mathcal{C}_1$. Now suppose $A \in \mathcal{C}_1$, so $A \times (a_2, b_2) \times \cdots \times (a_d, b_d) \in \Sigma_{\text{prod}}$. Then

$$A^c \times (a_2, b_2) \times \cdots \times (a_d, b_d) = (A \times (a_2, b_2) \times \cdots \times (a_d, b_d))^c \cap (\mathbb{R} \times (a_2, b_2) \times \cdots \times (a_d, b_d)) \in \Sigma_{\text{prod}}$$

by closure under complements and intersection (since we showed already that $\mathbb{R} \times (a_2, b_2) \times \cdots \times (a_d, b_d) \in \Sigma_{\text{prod}}$). Hence $A^c \in \mathcal{C}_1$. Finally, suppose $A_i \in \mathcal{C}_1$ for all $i \in \mathbb{N}$. Then each $A_i \times (a_2, b_2) \times \cdots \times (a_d, b_d) \in \Sigma_{\text{prod}}$. Observe that

$$(\cup_{i=1}^{\infty} A_i) \times (a_2, b_2) \times \cdots \times (a_d, b_d) = \cup_{i=1}^{\infty} (A_i \times (a_2, b_2) \times \cdots \times (a_d, b_d)) \in \Sigma_{\text{prod}}$$

by closure under countable union. Thus $\cup_{i=1}^{\infty} A_i \in \mathcal{C}_1$, meaning $\mathcal{C}_1$ is a σ-algebra.

Now, $(a, b) \in \mathcal{C}_1$ for any $a < b$ by definition of Σ_{prod}. Since $\mathcal{C}_1$ is a σ-algebra, $\sigma(\{(a, b) \mid a_i < b_i\} \subseteq \mathcal{C}_1$; that is, the intersection of all σ-algebras containing each (a, b) is a subset of $\mathcal{C}_1$. However, by Exercise 1.1.17, $\sigma(\{(a, b)\} = \mathcal{B}_{\mathbb{R}}$. Then $A_1 \in \mathcal{B}_{\mathbb{R}} \subseteq \mathcal{C}_1$, so

$$A_1 \times (a_2, b_2) \times \cdots \times (a_d, b_d) \in \Sigma_{\text{prod}}.$$

Similarly, define

$$\mathcal{C}_2 = \{A \in \mathcal{B}_{\mathbb{R}} \mid A_1 \times A \times (a_3, b_3) \times \cdots \times (a_d, b_d) \in \Sigma_{\text{prod}}\}$$

and by an analogous argument, $\mathcal{C}_2$ is a σ-algebra. Since $(a, b) \in \mathcal{C}_2$, $\mathcal{B}_{\mathbb{R}} = \sigma(\{(a, b)\} \subseteq \mathcal{C}_2$. Since $A_2 \in \mathcal{B}_{\mathbb{R}}$, we have $A_2 \in \mathcal{C}_2$ and so

$$A_1 \times A_2 \times (a_3, b_3) \times \cdots \times (a_d, b_d) \in \Sigma_{\text{prod}}.$$

Repeating the argument for another $d - 2$ iterations yields that

$$A_1 \times \cdots \times A_d \in \Sigma_{\text{prod}}$$

as desired. Therefore, $\mathcal{B}_{\mathbb{R}} \times \cdots \times \mathcal{B}_{\mathbb{R}} = \sigma(\{A_1 \times \cdots \times A_d \mid A_i \in \mathcal{B}_{\mathbb{R}}\} \subseteq \Sigma_{\text{prod}}$.

For the reverse containment, note that any $(a_1, b_1) \times \cdots \times (a_d, b_d) \in \{A_1 \times \cdots \times A_d \mid A_i \in \mathcal{B}_{\mathbb{R}}\}$ since clearly each $(a_i, b_i) \in \mathcal{B}_{\mathbb{R}} = \sigma(\{(a, b) \mid a < b\})$. Thus, $(a_1, b_1) \times \cdots \times (a_d, b_d) \in \sigma(\{A_1 \times \cdots \times A_d \mid A_i \in \mathcal{B}_{\mathbb{R}}\})$ and so $\sigma(\{(a_1, b_1) \times \cdots \times (a_d, b_d) \mid a_i, b_i \in \mathbb{R}, a_i < b_i\}) \subseteq \sigma(\{A_1 \times \cdots \times A_d \mid A_i \in \mathcal{B}_{\mathbb{R}}\}) = \mathcal{B}_{\mathbb{R}} \times \cdots \times \mathcal{B}_{\mathbb{R}}$. Hence, $\sigma(\{(a_1, b_1) \times \cdots \times (a_d, b_d) \mid a_i, b_i \in \mathbb{R}, a_i < b_i\}) = \Sigma_{\text{prod}} \subseteq \mathcal{B}_{\mathbb{R}} \times \cdots \times \mathcal{B}_{\mathbb{R}}$. By mutual containment,

$$\Sigma_{\text{prod}} = \mathcal{B}_{\mathbb{R}} \times \cdots \times \mathcal{B}_{\mathbb{R}}.$$

We now prove that $\mathcal{B}_{\mathbb{R}^d} = \Sigma_{\text{prod}}$. By definition, $\mathcal{B}_{\mathbb{R}^d} = \sigma(\{O \subseteq \mathbb{R}^d \mid O \text{ open}\})$. By Lemma 1.1.21.b (following this problem), if $O \subseteq \mathbb{R}^d$ is open in $\mathbb{R}^d$ then

$$O = \cup_{i=1}^{\infty}(a_{1i}, b_{1i}) \times \cdots \times (a_{di}, b_{di})$$

for $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$; that is, every open set O in $\mathbb{R}^d$ is the union of *countably many* open boxes B of the form $B = (a_1, b_1) \times \cdots \times (a_d, b_d)$. Since each such $B \in \Sigma_{\text{prod}}$ and Σ_{prod} is closed under countable union, every $O \in \Sigma_{\text{prod}}$ as well. Thus, $\mathcal{B}_{\mathbb{R}^d} = \sigma(\{O \subseteq \mathbb{R}^d \mid O \text{ open}\}) \subseteq \Sigma_{\text{prod}}$.

Since each $(a_1, b_1) \times \cdots \times (a_d, b_d)$ is open in $\mathbb{R}^d$, $(a_1, b_1) \times \cdots \times (a_d, b_d) \in \sigma(\{O \subseteq \mathbb{R}^d \mid O \text{ open}\}) = \mathcal{B}_{\mathbb{R}^d}$. As a result, $\sigma(\{(a_1, b_1) \times \cdots \times (a_d, b_d) \mid a_i < b_i\}) = \Sigma_{\text{prod}} \subseteq \mathcal{B}_{\mathbb{R}^d}$. By mutual containment, $\mathcal{B}_{\mathbb{R}^d} = \Sigma_{\text{prod}}$. Therefore, we have shown overall that $\mathcal{B}_{\mathbb{R}^d} = \Sigma_{\text{prod}} = \mathcal{B}_{\mathbb{R}} \times \cdots \times \mathcal{B}_{\mathbb{R}}$, or equivalently

$$\mathcal{B}_{\mathbb{R}^d} = \mathcal{B}_{\mathbb{R}} \times \cdots \times \mathcal{B}_{\mathbb{R}} = \sigma(\{(a_1, b_1) \times \cdots \times (a_d, b_d) \mid a_i, b_i \in \mathbb{R}, a_i < b_i\}).$$

□

Lemma 1.1.21.a. Let O be an open set in $\mathbb{R}^d$. Let $\|\cdot\|$ be any norm on $\mathbb{R}^d$. Let $B_r(x) = \{y \in \mathbb{R}^d \mid \|x - y\| < r\}$ denote the open ball of radius r centered at $x \in \mathbb{R}^d$. Then O is the union of countably many open "balls". Specifically, let $x \mapsto r(x)$ be a well-defined function that assigns each x exactly one radius $r(x)$. Then

$$O = \cup_{x \in C} B_{r(x)}(x) \qquad (C \text{ countable})$$

for some countable set C.

Proof. First suppose $O \neq \mathbb{R}^d$, so there exists some $w \in \mathbb{R}^d$ such that $w \notin O$. Let $x \in O$. By definition of open, there exists an $r > 0$ such that $B_r(x) \subset O$. Therefore, the set

$$\{r > 0 \mid B_r(x) \subset O\}$$

is nonempty. Since $O \neq \mathbb{R}^d$, this set is bounded above; in particular, for any $r > \|x - w\|$ we have $w \in B_r(x) \not\subseteq O$ since $w \notin O$. So $B_r(x) \subset O$ implies that $r \leq \|x - w\|$. Thus, $\{r > 0 \mid B_r(x) \subset O\}$ is a nonempty subset of $\mathbb{R}$ that is bounded above, so its supremum exists and is unique. Consequently, we can define

$$r_x := \sup\{r > 0 \mid B_r(x) \subset O\}.$$

Then $B_{r_x}(x) \subseteq O$. (If not, then there exists some $y \in B_{r_x}(x)$ such that $y \notin O$. Then for any $r' \in \mathbb{R}$ such that $\|x - y\| < r' < r_x$ we have $y \in B_{r'}(x)$ and so $B_{r'}(x) \not\subseteq O$; thus, $\|x - y\|$ is an upper bound of $\{r > 0 \mid B_r(x) \subset O\}$ with $r' < r_x$. This contradicts that the supremum r_x is the least upper bound.

Since $x \in O$ was arbitrary, $B_{r_x}(x) \subseteq O$ for any $x \in O$. So we have $\cup_{x \in O} B_{r_x}(x) \subseteq O$. Clearly $O \subseteq \cup_{x \in O} B_{r_x}(x)$ and so

$$O = \cup_{x \in O} B_{r_x}(x)$$

by mutual containment. However, the above union may not be countable. Let $\widetilde{Q} = O \cap \mathbb{Q}^d$. We will then show that

$$O = \bigcup_{q \in \widetilde{Q}} B_{r_q}(q).$$

Since $\mathbb{Q}^d$ is countable and $\widetilde{Q} = O \cap \mathbb{Q}^d \subset \mathbb{Q}^d$, we have that this union over $\widetilde{Q}$ is countable as well.

Let $x \in O$. By density of $\mathbb{Q}^d$ in $\mathbb{R}^d$, for every $\epsilon > 0$ there exists some $q \in \mathbb{Q}^d$ such that $x \in B_\epsilon(q)$. Take $\epsilon = \frac{r_x}{2}$. Then for $y \in B_\epsilon(q)$,

$$\|x - y\| \leq \|x - q\| + \|q - y\| < \tfrac{r_x}{2} + \tfrac{r_x}{2} = r_x$$

meaning $y \in B_{r_x}(x)$. Thus, $B_\epsilon(q) \subset B_{r_x}(x) \subseteq O$ with $x \in B_\epsilon(q)$. Note that since $B_\epsilon(q) \subset O$, we have that $q \in O \cap \mathbb{Q}^d = \widetilde{Q}$. Furthermore, we see that $\epsilon \in \{r > 0 \mid B_r(q) \subset O\}$, meaning by definition of r_q (the supremum) $\epsilon < r_q$. Hence, $B_\epsilon(q) \subseteq B_{r_q}(q)$. Since $x \in B_\epsilon(q)$ we have $x \in B_{r_q}(q) \subseteq \bigcup_{q \in \widetilde{Q}} B_{r_q}(q)$. Since $x \in O$ was arbitrary,

$$O \subseteq \bigcup_{q \in \widetilde{Q}} B_{r_q}(q).$$

For the reverse containment, note that each $B_{r_q}(q) \subseteq O$ by construction (as shown previously), so $\bigcup_{q \in \widetilde{Q}} B_{r_q}(q) \subseteq O$. Then

$$O = \bigcup_{q \in \widetilde{Q}} B_{r_q}(q).$$

as desired.

For the remaining case, suppose $O = \mathbb{R}^d$. For each $q \in \mathbb{Q}^d$, let its corresponding radius $r(q) = 1$. Clearly $\cup_{q \in \mathbb{Q}^d} B_1(q) \subseteq \mathbb{R}^d = O$. By density of $\mathbb{Q}^d$ in $\mathbb{R}^d$, for each $x \in \mathbb{R}^d$ there exists some $q' \in \mathbb{Q}^d$ such that $x \in B_1(q') \subset \cup_{q \in \mathbb{Q}^d} B_1(q)$, so $O \subseteq \cup_{q \in \mathbb{Q}^d} B_1(q)$. Thus, $O = \cup_{q \in \mathbb{Q}^d} B_1(q)$, which is a countable union of open balls. □

Lemma 1.1.21.b. Let O be an open set in $\mathbb{R}^d$. Then

$$O = \cup_{j=1}^{\infty} B_j, \qquad B_j = (a_{1j}, b_{1j}) \times \cdots \times (a_{dj}, b_{dj})$$

for some $a_{ij}, b_{ij} \in \mathbb{R}$, where $a_{ij} < b_{ij}$ for all $i \in \{1, \ldots, d\}, j \in \mathbb{N}$.

Proof. From the lemma above, we have $O = \cup_{q \in O \cap \mathbb{Q}^d} B_{r(q)}(q)$ for some radius function $r(q)$. However, since this equality holds for any norm, we can choose the *max* norm $\|x\|_\infty = \max_{i=1,\ldots,d} |x_i|$. For this norm, each "ball" is an open box:

$$B_r(q) = \{x \in \mathbb{R}^d \mid \max_j |x_j - q_j| < r\} = (q_1 - r, q_1 + r) \times \cdots \times (q_d - r, q_d + r).$$

Since $O \cap \mathbb{Q}^d$ is countable, we can enumerate its elements. For the jth $q \in O \cap \mathbb{Q}^d$, let $a_{ij} = q_i - r$ and $b_{ij} = q_i + r$. Thus, the jth $q \in O \cap \mathbb{Q}^d$ has corresponding box $B_j \equiv B_{r(q)}(q) = (a_{1j}, b_{1j}) \times \cdots \times (a_{dj}, b_{dj})$. Hence,

$$O = \cup_{j=1}^{\infty} B_j = \cup_{j=1}^{\infty} \left[(a_{1j}, b_{1j}) \times \cdots \times (a_{dj}, b_{dj})\right]$$

for $a_{ij}, b_{ij} \in \mathbb{R}$. □

Question 1.1.22. Let Ω be the sample space. For $A_\alpha \in 2^\Omega$ let $\mathcal{F} = \sigma(\{A_\alpha\}_{\alpha \in \Gamma})$, where the index set Γ is uncountable. Let $B \in \mathcal{F}$. Then there exists a countable sub-collection of indices $\{\alpha_j\}_{j=1}^{\infty} \subset \Gamma$ such that $B \in \sigma(\{A_{\alpha_j}\}_{j=1}^{\infty})$.[3]

[3]The notation $\{\alpha_j\}_{j=1}^{\infty} \subset \Gamma$ seems to suggest that $\{\alpha_j\}$ is countable and infinite; however, of course $\{\alpha_j\}$ can be finite (thus countable) throughout the proof.

Proof. Let

$$\mathcal{C} = \{A \in \mathcal{F} \mid A \in \sigma(\{A_{\alpha_j}\}_{j=1}^{\infty}) \text{ for some } \{\alpha_j\} \subset \Gamma\}.$$

For every $\alpha \in \Gamma$, $A_\alpha \in \sigma(\{A_\alpha\})$; that is, every A_α is contained in the σ-algebra generated by the "collection" containing only the set A_α. Since the collection of one set $\{A_\alpha\}$ clearly has countably many "elements" (i.e. one element, the set A_α), each $A_\alpha \in \mathcal{C}$. Since $\mathcal{C}$ is a σ-algebra**, $\mathcal{C}$ is a σ-algebra containing $\{A_\alpha\}_{\alpha\in\Gamma}$, and therefore $\mathcal{F} = \sigma(\{A_\alpha\}_{\alpha\in\Gamma}) \subseteq \mathcal{C}$. Hence, $B \in \mathcal{F}$ implies that $B \in \mathcal{C}$. Therefore, $B \in \sigma(\{A_{\alpha_j}\}_{j=1}^{\infty})$ for some countable $\{\alpha_j\} \subset \Gamma$. □

*Proof (of **).* Fix $\alpha \in \Gamma$. Then $\{\alpha\} \subset \Gamma$ is finite (and thus countable). Since $\sigma(\{A_\alpha\})$ is a σ-algebra, $\Omega, \emptyset \in \sigma(\{A_\alpha\})$. Hence, $\Omega, \emptyset \in \mathcal{C}$. Now suppose $A \in \mathcal{C}$, so $A \in \sigma(\{A_{\alpha_j}\})$ for some countable $\{\alpha_j\} \subset \Gamma$. Since $\sigma(\{A_{\alpha_j}\})$ is a σ-algebra (and thus closed under complements), $A^c \in \sigma(\{A_{\alpha_j}\})$ and thus $A^c \in \mathcal{C}$.

Finally, let $A_i \in \mathcal{C}$ for all $i \in \mathbb{N}$. Then each $A_i \in \sigma(\mathcal{A}_i)$ where $\mathcal{A}_i$ is a countable subset of $\{A_\alpha\}_{\alpha\in\Gamma}$. Since each $\mathcal{A}_i$ has countably many "elements" (i.e. sets), $\cup_{i=1}^{\infty}\mathcal{A}_i$ is also a countable subset of $\{A_\alpha\}_{\alpha\in\Gamma}$. From $\mathcal{A}_i \subseteq \cup_{i=1}^{\infty}\mathcal{A}_i$, each $\sigma(\mathcal{A}_i) \subseteq \sigma(\cup_{i=1}^{\infty}\mathcal{A}_i)$ and therefore $A_i \in \sigma(\mathcal{A}_i) \subseteq \sigma(\cup_{i=1}^{\infty}\mathcal{A}_i)$. Since $\sigma(\cup_{i=1}^{\infty}\mathcal{A}_i)$ is a σ-algebra containing each set A_i, by closure under countable union, $\cup_{i=1}^{\infty}A_i \in \sigma(\cup_{i=1}^{\infty}\mathcal{A}_i)$. Thus, $\cup_{i=1}^{\infty}A_i$ is contained in a countably generated σ-algebra whose generators are a subset of $\{A_\alpha\}_{\alpha\in\Gamma}$. Therefore, $\cup_{i=1}^{\infty}A_i \in \mathcal{C}$. We conclude that $\mathcal{C}$ is a σ-algebra.

Question 1.1.33 Let $(\Omega, \mathcal{F})$ be a probability space. Let $\mathcal{A} \subseteq 2^\Omega$ and $\mathcal{F} = \sigma(A)$. Show that for some $\Omega, \mathcal{A}$ there are two probability measures $\mu \neq \nu$ such that $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$.

Proof. As suggested by the hint, let $\Omega = \{1, 2, 3, 4\}$. Let $\mathcal{A} = \{\{1, 2\}, \{1, 3\}\}$ (which is *not* an algebra). Then $\mathcal{F} = \sigma(\mathcal{A}) = 2^\Omega$. Define $\mu, \nu : \mathcal{F} \to [0, 1]$ by

$$\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = \mu(\{4\}) = \tfrac{1}{4}$$
$$\nu(\{1\}) = \nu(\{4\}) = \tfrac{1}{6}, \nu(\{2\}) = \nu(\{3\}) = \tfrac{1}{3}.$$

Then $\mu(\{1, 2\}) = \mu(\{1\}) + \mu(\{2\}) = \frac{1}{2}$ and $\mu(\{1, 3\}) = \frac{1}{2}$. Also, $\nu(\{1, 2\}) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ and similarly $\nu(\{1, 3\}) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$. Thus, $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$ but $\mu \neq \nu$. □