MATHEMATICS 210A: HOMEWORK 1

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For all exercises, assume that X is the underlying topological space and $I = [0,1]$ unless otherwise specified.

Hatcher Section 1.1.

(1) Show that the composition of paths satisfies the following cancellation property: If $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$ then $f_0 \simeq f_1$.

Proof. First we prove the following lemma about inverse paths.

Lemma. Let $f, g : I \to X$ be paths such that $f \simeq g$. Let $\bar{f}, \bar{g} : I \to X$ be the corresponding *inverse paths* as defined in Hatcher in the proof of Proposition 1.3; that is,

$$\begin{aligned} \bar{f} : I \to X, \ \ \bar{f}(s) &= f(1-s) \\ \bar{g} : I \to X, \ \ \bar{g}(s) &= g(1-s). \end{aligned}$$

Then $\bar{f} \simeq \bar{g}$ as well.

Proof. Let $H : I \times I \to X$ be a homotopy between f and g, so $H(s,0) = f(s)$ and $H(s,1) = g(s)$. Then consider the map

$$\overline{H} : I \times I \to X \text{ s.t. } \overline{H}(s,t) = H(1-s,t).$$

Then $\overline{H}$ is continuous since H is continuous (since $\overline{H}$ is the composition of continuous functions). Furthermore, $\overline{H}(s,0) = H(1-s,0) = f(1-s) = \bar{f}(s)$ and similarly $\overline{H}(s,1) = \bar{g}(s)$. Letting $H_t = H(\ \cdot\ ,t)$, we have that (since $f \overset{H_t}{\simeq} g$) $H_t(0) = x_0$ and $H_t(1) = x_1$ for some $x_0, x_1 \in X$ independent of t. Thus

$$\overline{H}(0,t) = H(1,t) = H_t(1) = x_1 \text{ and } \overline{H}(1,t) = H(0,t) = H_t(0) = x_0$$

meaning the endpoints of $\overline{H}(s,t)$ are fixed in time. Hence $\overline{H}(\ \cdot\ ,t)$ is a homotopy between $\bar{f}$ and $\bar{g}$, so $\bar{f} \simeq \bar{g}$. □

Now suppose $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ for paths $f_0, f_1, g_0, g_1 : I \to X$ with $f_0(1) = g_0(0), f_1(1) = g_1(0)$ (so the path compositions are defined). Furthermore, suppose $g_0 \simeq g_1$. Consider the inverse paths $\overline{g_0}, \overline{g_1}$. Since $g_0 \simeq g_1$, by the lemma above, $\overline{g_0} \simeq \overline{g_1}$.

Let $c : I \to X$ be the constant path such that $c(s) = f_0(1)$. Then, as seen in the proof of Proposition 1.3, $f_0 \cdot c$ is a reparametrization of f_0, and thus $f_0 \simeq f_0 \cdot c$. Similarly, $f_1 \simeq f_1 \cdot d$ where $d(s) = f_1(1)$ is a constant path. Furthermore, within the same proof, it is shown that $g_0 \cdot \overline{g_0} \simeq \tilde{c}$ where $\tilde{c}(s) = g_0(0)$ for all $s \in I$. However, since $g_0(0) = f_0(1)$, we have that $c = \tilde{c}$. Similarly $d = \tilde{d}$ where $g_1 \cdot \overline{g_1} \simeq \tilde{d}$. So by substituting, and by the symmetry and transitivity of $\simeq$, we have that

$$\begin{aligned} f_0 &\simeq f_0 \cdot c = f_0 \cdot \tilde{c} \\ &\simeq f_0 \cdot (g_0 \cdot \overline{g_0}) && [\text{since } g_0 \cdot \overline{g_0} \simeq \tilde{c}] \\ &\simeq (f_0 \cdot g_0) \cdot \overline{g_0} && [\text{path composition is associative}] \\ &\simeq (f_1 \cdot g_1) \cdot \overline{g_0} && [\text{since } f_0 \cdot g_0 \simeq f_1 \cdot g_1] \\ &\simeq (f_1 \cdot g_1) \cdot \overline{g_1} && [\text{since } \overline{g_0} \simeq \overline{g_1}] \\ &\simeq f_1 \cdot (g_1 \cdot \overline{g_1}) \\ &\simeq f_1 \cdot \tilde{d} = f_1 \cdot d \simeq f_1 \end{aligned}$$

which concludes the proof. □

(2) Show that change-of-basepoint homomorphism β_h depends only on the homotopy class of h.

Proof. The change-of-basepoint homomorphism $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ is defined for a path $h : I \to X$ by

$$\beta_h[f] = [h \cdot f \cdot \bar{h}].$$

Suppose $h_1 \simeq h_2$ for paths h_1, h_2. By the lemma in problem (1), $\overline{h_1} \simeq \overline{h_2}$; that is, the inverse paths are homotopic. Then for any $[f] \in \pi_1(X, x_1)$, $f \cdot \overline{h_1} \simeq f \cdot \overline{h_2}$. Since $h_1 \simeq h_2$, we then have that

$$h_1 \cdot (f \cdot \overline{h_1}) \simeq h_2 \cdot (f \cdot \overline{h_2}).$$

So $h_1 \cdot f \cdot \overline{h_1}$ and $h_2 \cdot f \cdot \overline{h_2}$ are in the same homotopy class, meaning $[h_1 \cdot f \cdot \overline{h_1}] = [h_2 \cdot f \cdot \overline{h_2}]$. Therefore,

$$\beta_{h_1}[f] = [h_1 \cdot f \cdot \overline{h_1}] = [h_2 \cdot f \cdot \overline{h_2}] = \beta_{h_2}[f]$$

when $h_1 \simeq h_2$. Hence, β_h only depends on the homotopy class of h. □

(10) From the isomorphism $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ it follows that loops in $X \times \{y_0\}$ and $\{x_0\} \times Y$ represent commuting elements of $\pi_1(X \times Y, (x_0, y_0))$. Construct an explicit homotopy demonstrating this.

Proof. Let $\ell : I \to X \times \{y_0\}$ and $\ell' : I \to \{x_0\} \times Y$ be loops each based at the point (x_0, y_0). Then their composition $\ell' \cdot \ell$ is defined since $\ell(0) = (x_0, y_0) = \ell'(1)$. Define the projections of ℓ and ℓ' onto their components in X and Y respectively; that is, for any $s \in I$ the loops $\ell_{\text{proj}}(s)$ and $\ell'_{\text{proj}}(s)$ are defined such that

$$\ell(s) = (\ell_{\text{proj}}(s), y_0) \text{ and } \ell'(s) = (x_0, \ell'_{\text{proj}}(s)).$$

Consider the maps

$$f : I \times I \to X \text{ s.t. } f(s,t) = \begin{cases} x_0 & s \in [0, \frac{t}{2}] \\ \ell_{\text{proj}}(2s - t) & s \in [\frac{t}{2}, \frac{1+t}{2}] \\ x_0 & s \in [\frac{1+t}{2}, 1] \end{cases}$$

$$g : I \times I \to Y \text{ s.t. } g(s,t) = \begin{cases} y_0 & s \in [0, \frac{1-t}{2}] \\ \ell'_{\text{proj}}(2s + t - 1) & s \in [\frac{1-t}{2}, \frac{2-t}{2}] \\ y_0 & s \in [\frac{2-t}{2}, 1]. \end{cases}$$

First we verify that f is continuous by showing that $f(s, \cdot)$ is continuous for any fixed $s \in I$ and $f(\cdot, t)$ is continuous for any fixed $t \in I$. Clearly $f(\cdot, t)$ is continuous on $[0, \frac{t}{2}]$ and $[\frac{1+t}{2}, 1]$ since it is constant on those intervals. On $[\frac{t}{2}, \frac{1+t}{2}]$, $f(\cdot, t)$ is also continuous since it is the composition of continuous functions (ℓ_{proj} is a path and thus continuous). At the first boundary point, $f(\frac{t}{2}, t) = x_0 = \ell_{\text{proj}}(0) = \ell_{\text{proj}}(2(\frac{t}{2}) - t)$ since $\ell(0) = (x_0, y_0)$. In addition for $s = \frac{1+t}{2}$, $\ell_{\text{proj}}(2s - t) = \ell_{\text{proj}}(1) = x_0$. Therefore, $f(\cdot, t)$ is continuous on I.

Similarly, fix $s \in I$ and consider $f(s, \cdot)$. Then any input $t \in I$ satisfies at least one of the following: $0 \le t \le 2s - 1, 2s - 1 \le t \le 2s$, or $2s \le t \le 1$. For this first interval, $f(s,t) = x_0$ since $t \le 2s - 1 \Leftrightarrow \frac{t+1}{2} \le s \le 1$ (since $s \in [0,1]$) $\Leftrightarrow s \in [\frac{1+t}{2}, 1]$. Similarly, $f(s,t) = x_0$ on the last interval. For $2s - 1 \le t \le 2s$, $s \le \frac{1+t}{2}$ (first inequality) and $s \ge \frac{t}{2}$ (second inequality), so $f(s,t) = \ell_{\text{proj}}(2s - t)$ on this interval of t, which is continuous. Finally, for the boundary points $t = 2s - 1$ and $t = 2s$, we have that $f(s,t) = \ell_{\text{proj}}(2s - (2s - 1)) = \ell_{\text{proj}}(1) = x_0$ and $f(s,t) = \ell_{\text{proj}}(2s - 2s) = \ell_{\text{proj}}(0) = x_0$ respectively. Hence, $f(s, \cdot)$ is continuous as well. Hence f is continuous on $I \times I$.

By an analogous argument, g is continuous on $I \times I$. Thus the proposed homotopy

$$F : I \times I \to X \text{ s.t. } F(s,t) = (f(s,t), g(s,t))$$

is continuous. Also, the endpoints of F are fixed in time since

$$F(0,t) = (f(0,t), g(0,t)) = (x_0, y_0) \text{ and } F(1,t) = (f(1,t), g(1,t)) = (x_0, y_0).$$

Finally, by definition of path composition

$$\ell \cdot \ell'(s) = \begin{cases} \ell(2s) & s \in [0, \frac{1}{2}] \\ \ell'(2s - 1) & s \in [\frac{1}{2}, 1] \end{cases} = \begin{cases} (\ell_{\text{proj}}(2s), y_0) & s \in [0, \frac{1}{2}] \\ (x_0, \ell'_{\text{proj}}(2s - 1)) & s \in [\frac{1}{2}, 1] \end{cases}$$

and similarly

$$\ell' \cdot \ell(s) = \begin{cases} (x_0, \ell'_{\text{proj}}(2s)) & s \in [0, \frac{1}{2}] \\ (\ell_{\text{proj}}(2s-1), y_0) & s \in [\frac{1}{2}, 1]. \end{cases}$$

Therefore,

$$F(s,0) = (f(s,0), g(s,0)) = \begin{cases} (\ell_{\text{proj}}(2s-0), y_0) & s \in [0, \frac{1}{2}] \\ (x_0, \ell'_{\text{proj}}(2s+0-1)) & s \in [\frac{1}{2}, 1] \end{cases} = \ell \cdot \ell'(s)$$

and

$$F(s,1) = (f(s,1), g(s,1)) = \begin{cases} (x_0, \ell'_{\text{proj}}(2s+1-1)) & s \in [0, \frac{1}{2}] \\ (\ell_{\text{proj}}(2s-1), y_0) & s \in [\frac{1}{2}, 1] \end{cases} = \ell' \cdot \ell(s).$$

Hence $\ell \cdot \ell' \overset{F}{\simeq} \ell' \cdot \ell$, meaning loops in $X \times \{y_0\}$ and $\{x_0\} \times Y$ represent commuting elements of $\pi_1(X \times Y, (x_0, y_0))$. □

(12) Show that every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism φ_* of a map $\varphi : S^1 \to S^1$.

Proof. Without loss of generality, embed $S^1 \hookrightarrow \mathbb{C}$ in the complex plane. Let w_n be a loop in S^1 based at $1_{\mathbb{C}}$ defined by

$$w_n : I \to S^1 \text{ s.t. } \ell(s) = e^{2\pi i n s}.$$

In the proof of $\pi_1(S^1) \cong \mathbb{Z}$ (Theorem 1.7), the map $\Phi : \mathbb{Z} \to \pi_1(S^1)$ defined by $\Phi(n) = [w_n]$ was shown to be an isomorphism. Now any homomorphism $\Psi : \mathbb{Z} \to \mathbb{Z}$ has the form $\Psi(z) = n \cdot z$ with $n \in \mathbb{Z}$. Then by the commutative diagram, observe that $\Phi(1_{\mathbb{Z}}) = [w_1]$ and $\Phi \circ \Psi(1_{\mathbb{Z}}) = \Phi(n) = [w_n]$, meaning any homomorphism $f : \pi_1(S^1) \to \pi_1(S^1)$ sends $[w_1] \mapsto [w_n]$. Furthermore, since $1_{\mathbb{Z}}$ is a generator of $\mathbb{Z}$, by the group isomorphism $[w_1]$ is the generator of $\pi_1(S^1)$.

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow[1 \mapsto n \cdot 1]{} & \mathbb{Z} \\ \Phi \updownarrow \simeq & & \Phi \updownarrow \simeq \\ \pi_1(S^1) & \xrightarrow[{[w_1] \mapsto [w_n]}]{} & \pi_1(S^1). \end{array}$$

Consider the map $\varphi : S^1 \to S^1$ such that $\varphi(z) = z^n$. Then φ induces the map

$$\varphi_* : \pi_1(S^1) \to \pi_1(S^1) \text{ s.t. } \varphi_*([w]) = [\varphi w]$$

which is determined by its image on generator $[w_1]$:

$$\varphi_*([\omega_1]) = [\varphi \circ (s \mapsto w_1(s))] = [s \mapsto (e^{2\pi i s})^n] = [w_n] = f([\omega_1]).$$

Thus, $f = \varphi_*$ since these maps are identical on generator $[w_1]$ for f an arbitrary homomorphism and $\varphi(z) = z^n$, where n is the number of times f wraps a loop around S^1. Hence, every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism φ_* of a map $\varphi : S^1 \to S^1$.

□

(16) Show there are no retractions $r : X \to A$ in the following cases:

(a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1
(b) $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$
(c) $X = S^1 \times D^2$ with A the circle shown in the figure
(d) $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$
(e) X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$
(f) X the Möbius band and A its boundary circle

Proof. For each case, we suppose for contradiction that X retracts onto subspace A. Then by Proposition 1.17, the homomorphism $\iota_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $\iota : A \hookrightarrow X$ is injective. Hence, if we show that $\iota_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ cannot be injective, we will have shown that there are no retractions $r : X \to A$ in that case.

(a) By Proposition 1.14, $\pi_1(X) = \pi_1(\mathbb{R}^3) \cong 0$. However, since A is homeomorphic to S^1 and $\pi_1(S^1) \cong \mathbb{Z}$ (Theorem 1.7), we have that $\pi_1(A) \cong \pi_1(S^1) \cong \mathbb{Z}$. So the homomorphism

$$\iota_* : \mathbb{Z} \to 0$$

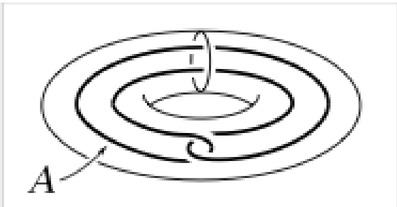
is injective. This is a contradiction since $\mathbb{Z}$ cannot inject homomorphically into 0.

(b) Since S^1 and D^2 are path-connected, by Proposition 1.12, $\pi_1(S^1 \times D^2) \cong \pi_1(S^1) \times \pi_1(D^2)$. However, by the proof of Theorem 1.9 (Brouwer Fixed Point Theorem), D^2 is contractible i.e. $\pi_1(D^2) \cong 0$. By Theorem 1.7, $\pi_1(S^1) \cong \mathbb{Z}$. Thus $\pi_1(S^1) \times \pi_1(D^2) \cong \mathbb{Z} \times 0 \cong \mathbb{Z}$. Similarly, $\pi_1(A) = \pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$. So the homomorphism

$$\iota_* : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$$

is injective, which is impossible since $\mathbb{Z} \times \mathbb{Z}$ does not embed homomorphically into $\mathbb{Z}$. Hence, there are no retractions $r : X \to A$ in this case.

(c) Here A is the circle depicted in the figure below.



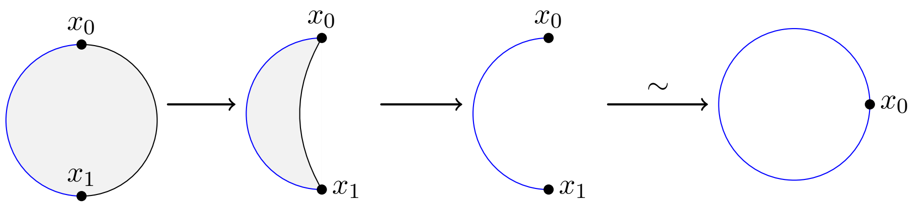
As in part (b), $\pi_1(X) \cong \mathbb{Z}$. From the figure we see (since A crosses itself in the front) that A is homeomorphic to S^1. Though by tracing A, we see that the homotopy class of any generating loop in $\pi_1(A)$ is sent by ι_* to a homotopy class of a loop in $\pi_1(X)$ that travels once around the S^1 factor and then (when reaching the crossing in A) travels again around S^1 in the *opposite* direction — which is homotopic to a trivial loop. Hence, the inclusion $\iota : A \hookrightarrow X$ induces a *trivial* map $\iota_* : \pi_1(A) \to \pi_1(X)$. Thus, ι_* not injective, meaning there are no retractions $r : X \to A$ in this case.

(d) For $A = S^1 \vee S^1$, by Section 1.2 and Van Kampen's Theorem, $\pi_1(A) = \pi_1(S^1 \vee S^1)$ is the free group on two generators F_2. However, since D^2 is contractible, so is $D^2 \vee D^2$ by contracting to the identified point. Thus $\pi_1(X) = \pi_1(D^2 \vee D^2) \cong 0$. So

$$\iota_* : F_2 \to \pi_1(X) \cong 0$$

is injective, which is a contradiction. Hence, there are no retractions $r : X \to A$ for $X = D^2 \vee D^2$ and $A = S^1 \vee S^1$.

(e) Let x_0, x_1 be points on the boundary of D^2. Consider an arc on D^2 from x_0 to x_1, as drawn in blue in the first figure below. Then D^2 can retract onto the blue arc, as seen in the first three figures, meaning D^2 is homotopic to this arc. From there, X is formed by identifying $x_0 \sim x_1$, as depicted in the last diagram. Hence, we see that $\pi_1(X) \cong \pi_1(S^1) \cong \mathbb{Z}$ (unlike in part (d) where $\pi_1(X) \cong 0$).



Since $A = S^1 \vee S^1$, again by Van Kampen's Theorem, we have that $\pi_1(A)$ is the free group on two generators (call them a and b). So we have the induced map

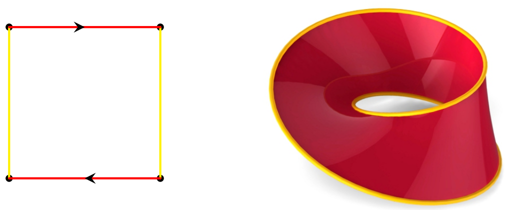
$$\iota_* : F_2 \hookrightarrow \mathbb{Z}.$$

Define $\iota_*(a) := m$ and $\iota_*(b) = n$. Then note that a^n and b^m are distinct elements in F_2. However, by definition of a group homomorphism (and since $\mathbb{Z}$ is abelian),

$$\iota_*(a^n) = \iota_*(a)n = mn = nm = \iota_*(b)m = \iota_*(b^m)$$

contradicting that ι_* is injective.

(f) Let M be the Möbius band. Consider the gluing diagram of M depicted below. The sides in red are glued, and the boundary circle A consists of the sides in gold.



From the figure, observe that M retracts onto the boundary circle A, meaning $\pi_1(M) \cong \pi_1(A) \cong \pi_1(S^1) \cong \mathbb{Z}$. So the induced map $\iota_* : \mathbb{Z} \to \mathbb{Z}$ is a homomorphism on $\mathbb{Z}$ and therefore has the form $\iota_*(z) = nz$ for some $n \in \mathbb{Z}$. From the second image above, we can see that a loop that goes once around the boundary circle A travels twice around the strip M. Hence, $\iota_*(z) = 2z$ for $z \in \mathbb{Z}$.

Now, if a retraction $r : M \to A$ existed, since $r_*\iota_* = \mathbb{1}_*$, $r_*\iota_*(1) = 1$. However, $\iota_*(1) = 2$, meaning $r_*(2) = 1$. Though, since r_* is a homomorphism, $r_*(2) = r_*(1+1) = r_*(1)+r_*(1) = 2r_*(1)$, and so $r_*(1) = \frac{1}{2}$. Hence $r_*(1) \notin \mathbb{Z}$, which is a contradiction. Thus the Mobius band does not retract onto its boundary circle.

□

(17) Construct infinitely many non-homotopic retractions $S^1 \vee S^1 \to S^1$.

Proof. For $S^1 \vee S^1$, denote the circles as S^1_1 and S^1_2 with S^1_2 as the rightmost circle. Without loss of generality, embed $S^1 \hookrightarrow \mathbb{C}$, and let the basepoint $x_0 \in X$ joining the circles be $x_0 = 0_{\mathbb{C}}$, so $S^1 \vee S^1$ is centered around the origin and the circles are centered around ± 1. (This proof will work for other values of x_0 by adjusting the coordinate system). Heuristically, each non-homotopic retraction will fix the rightmost S^1_2 and twist the leftmost S^1_1 around x_0 some integer number of times before projecting S^1_1 onto S^1_2.

Formally, define the map $f_n : S^1 \vee S^1 \to S^1$ by

$$f_n(z) = \begin{cases} z & \text{if } |z-1| = 1 \\ 1 + (z+1)^n & \text{if } |z+1| = 1 \end{cases}$$

for $n \in \mathbb{Z}$. Then f_n is clearly continuous and $f_n|_{S^1_2} = \mathbb{1}_{S^1_2}$ since f_n fixes points satisfying $|z-1| = 1$, meaning f_n fixes S^1_2. Hence, f_n is a retraction from $S^1 \vee S^1$ onto S^1.

We now show that the $\{f_n\}_{n\in\mathbb{Z}}$ are non-homotopic. Let $n_1, n_2 \in \mathbb{Z}$ with $n_1 \neq n_2$. Suppose $f_{n_1} \simeq f_{n_2}$. Now $\{f_n\}_{n\in\mathbb{Z}} \cong \mathbb{Z} \cong \pi_1(S^1)$ by mapping each $f_n \mapsto n \mapsto [w_n]$. However, since $f_{n_1} \simeq f_{n_2}$, this forces $[w_{n_1}] = [w_{n_2}]$ (which is a contradiction). Hence, the $\{f_n\}_{n\in\mathbb{Z}}$ produce infinitely many non-homotopic retractions.

□

Hatcher 1.2

(6) Use Proposition 1.26 to show that the complement of a closed discrete subspace of $\mathbb{R}^n$ is simply connected if $n \geq 3$.

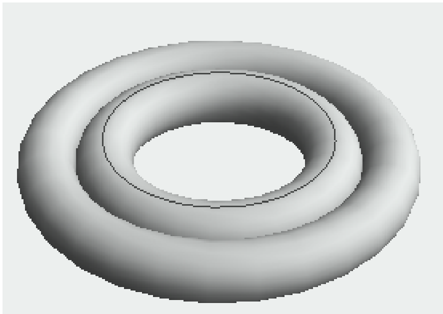
Proof. Let $S \subseteq \mathbb{R}^n$ be a closed discrete subspace of $\mathbb{R}^n$. Note that S is countable, since closed discrete subspaces of separable normal subsets are countable. Now, $\mathbb{R}^n - S$ deformation retracts onto a wedge sum of spheres $\bigvee_\alpha S^{n-1}$ for index set α with one $(n-1)$ sphere per element of S (so there are at most countably many summands). We can construct $\bigvee_\alpha S^{n-1}$ by attaching n-cells (one per S^{n-1}) to a common basepoint – and call this constructed space Y. So by Proposition 1.26,

$$\pi_1(R^n - S) \cong \pi_1(Y) \cong \pi_1\big(\bigvee_\alpha S^{n-1}\big) \cong 0 \text{ for } n \geq 3$$

where the third isomorphism follows from Proposition 1.14 (which says that $\pi_1(S^n) \cong 0$ for $n \geq 2$) and Van Kampen's Theorem. Furthermore, a topological space that is path-connected and has trivial fundamental group is said to be *simply connected*. Since $\mathbb{R}^n - S$ is clearly path-connected and $\pi_1(R^n - S) = 0$, we conclude that $R^n - S$ is simply connected. □

(8) Compute the fundamental group of the space X obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

Proof. Let $T_1, T_2 \subseteq X$ be the two tori identified along a circle $S^1 \times \{x_0\}$ in one and the corresponding $S^1 \times \{x_0\}$ in the other. Without loss of generality, assume this identification stacks the two tori on top of each other as depicted below.



Now, we know $\pi_1(T_1)$ is the free group on two generators (call them $a = [\ell_a]$ and $b = [\ell_b]$), and similarly $\pi_1(T_2) = \langle c, d\rangle$ for generating loops ℓ_c, ℓ_d in T_2. Label the generators such that the stacking depicted above identifies loops ℓ_a and ℓ_c.

Consider the neighborhoods N_1, N_2, where N_1 is a neighborhood of ℓ_a contained entirely in T_1 and N_2 is a neighborhood of ℓ_c contained entirely in T_2. Let $U = T_1 \cup N_2$ and $V = T_2 \cup N_1$, so U, V, and $U \cap V$ are all path-connected.

Then $U \cap V$ deformation retracts onto the circle ℓ_a (or equivalently ℓ_c), and hence $\pi_1(U \cap V) \cong \pi_1(S^1) \cong \mathbb{Z}$. Also, since U and V deformation retract onto T_1 and T_2 respectively (and $\pi_1(T_1) = \pi_1(T_2) \cong \mathbb{Z}^2$), we have $\pi_1(U), \pi_1(V) \cong \mathbb{Z}^2$. Let $[\ell_\Gamma]$ be a generator of $\pi_1(T_1 \cap T_2)$. Consider the maps induced by inclusion

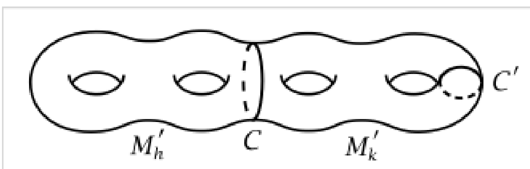
$$\iota_{1,*} : \pi_1(T_1 \cap T_2) \to \pi_1(T_1)$$
$$\iota_{2,*} : \pi_1(T_1 \cap T_2) \to \pi_1(T_2)$$

Then $\iota_{1,*}([\ell_\Gamma]) = [\ell_a] = a$ and $\iota_{2,*}([\ell_\Gamma]) = [\ell_c] = c$. By Van Kampen's theorem (since $X = U \cup V$), we have that $\pi_1(X) \cong (\pi_1(U) * \pi_1(V))/N$ where N is generated by $\iota_{1,*}([\ell_\Gamma])\iota_{2,*}([\ell_\Gamma])^{-1} = ac^{-1}$. Hence,

$$\begin{aligned}
\pi_1(X) &\cong \pi_1(U) * \pi_1(V)/N \\
&= \langle a, b, c, d : ab = ba, cd = dc\rangle/N \qquad \text{[since } \pi_1(U), \pi_1(V) \text{ are abelian]} \\
&\cong \langle a, b, c, d : ab = ba, cd = dc, a = c\rangle \\
&\cong \langle a, b, d : ab = ba, ad = da\rangle,
\end{aligned}$$

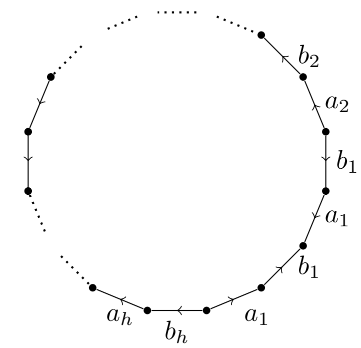
completing the calculation. Note that $\langle a, b, d : ab = ba, ad = da\rangle \cong \mathbb{Z} \times F$, where F is the free group on two generators. Hence, $\pi_1(X) \cong \langle a, b, d : ab = ba, ad = da\rangle \cong \mathbb{Z} \times F$. □

(9) In the surface M_g of genus g, let C be a circle that separates M_g into two compact subsurfaces M'_h and M'_k obtained from the closed surfaces M_h and M_k by deleting an open disk from each. Show that M'_h does not retract onto its boundary circle C, and hence M_g does not retract onto C. Also show that M_g *does* retract onto the non separating circle C' in the figure below.



Proof. Suppose for contradiction that there is a retraction $r : M'_h \to C$, so M'_h retracts onto its boundary circle C. Now the closed surface M_h (of genus h) has the following cell structure: one 0 cell, $2h$ 1 cells, and one 2 cell, similar to the example in Chapter 0 (page 5) of Hatcher.

The 1 cells correspond to (open arc) loops around every hole of M_h, each of which is then identified at the same basepoint h_0 (corresponding to the 0 cell). The 2 cell is an open $4h$-gon, which is depicted below.



This 2 cell provides a relation on $\pi_1(M_h)$ by traveling along the perimeter loop; that is, $\pi_1(M_h)$ has the presentation

$$\pi_1(M_h) = \langle a_1, b_1, \ldots, a_h, b_h | a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h b_h a_h^{-1} b_h^{-1} = 1 \rangle.$$

Let D_c be the open disk whose boundary is the circle C, so $M_h' = M_h \setminus D_c$. Without loss of generality, choose the representative loops in the $2h$ homotopy classes (corresponding to the 1 cells) such that they do not intersect $\overline{D_c}$; consequently, we can assume that $\overline{D_c}$ is contained entirely in the interior of the 2 cell.

Then M_h' has the same cell structure of M_h except that M_h' has an open circle removed from its 2 cell, meaning the perimeter of the 2 cell deformation retracts onto the boundary of this circle. Hence, the homotopy class of a generating loop around C corresponds to an element $[a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h b_h a_h^{-1} b_h^{-1}] \in \pi_1(M_h')$; that is, the induced map $\iota_* : \pi_1(C) \hookrightarrow \pi_1(M_h')$ sends the generator of $\pi_1(C) \cong \mathbb{Z}$ to the product of commutators $[a_1, b_1] \cdots [a_h, b_h]$ (similar to Example 2.36). However, this means that $\iota_* : \pi_1(C) \to \text{abel}(\pi_1(M_h'))$ is *trivial* since $\text{abel}(\pi_1(M_h'))$ quotients out this product of commutators. Consider the composition

$$\pi_1(C) \overset{\iota_*}{\hookrightarrow} \pi_1(M_h') \overset{r_*}{\to} \pi_1(C)$$

which is the identity map (since $r_* \iota_* = \mathbb{1}_*$). As the hint suggests, we abelianize the groups to yield the composition

$$\pi_1(C) \overset{\iota_*}{\hookrightarrow} \text{abel}(\pi_1(M_h')) \overset{r_*}{\to} \pi_1(C)$$

where $\text{abel}(\pi_1(C)) = \pi_1(C)$ since $\pi_1(C) \cong \mathbb{Z}$ is already abelian. Therefore, the above composition is the identity map with $\pi_1(C) \overset{\iota_*}{\hookrightarrow} \text{abel}(\pi_1(M_h'))$ trivial, which is impossible for group homomorphisms. Hence, M_h' does not retract onto C. Since a retract $r : M_g \to C$ would yield a retract of M_h' to C by restriction, we conclude that M_g does not retract onto C.

We now prove that M_g does retract onto C'. Analogous to the diagram above, the $4g$-gon quotients to M_g with the attachment map whose gluing is indicated by the perimeter arrows. The circle C' corresponds to one edge in the perimeter; without loss of generality, identify C' with the edge b_g. Then we can first contract edges a_1 and a_1^{-1} to a point, contract b_1 and b_1^{-1}, and continue in this manner until only edge b_g is left (corresponding to C'). Formally speaking, M_g has CW structure of one 0 cell, $2g$ 1 cells $(a_1, b_1, \ldots, a_g, b_g)$, and one 2 cell (the open $4g$-gon). Hence, the 1 skeleton of M_g is $\bigvee_{k=1}^{g} (S^1_{a_k} \vee S^1_{b_k})$ with 2 cell the product of commutators $[a_1, b_1] \cdots [a_g, b_g]$. Let $q : M_g \to M_1$ be the map that quotients by $\bigvee_{k=1}^{g-1} (S^1_{a_k} \vee S^1_{b_k})$, which is clearly a retraction. Consider the map

$$r : S^1 \times S^1 \cong M_1 \longrightarrow \{y\} \times S^1 \cong C' \quad \text{such that} \quad r(x_1, x_2) = (y, x_2)$$

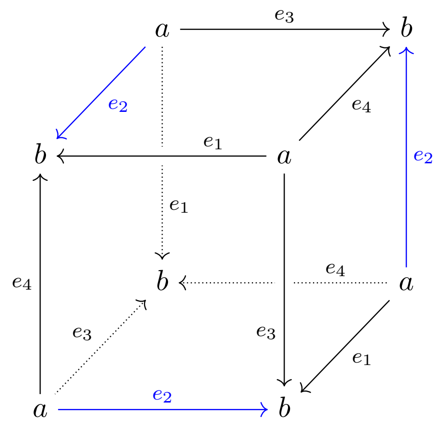
for some point $y \in S^1$. Then r is continuous, and (letting $(y, x_2) \in \{y\} \times S^1 \cong C'$ be arbitrary) $r|_{C'}(y, x_2) = (y, x_2)$, meaning $r|_{C'} = \text{id}_{C'}$. So r is a retraction as well. Hence the map

$$r \circ q : M_g \to C'$$

is a retraction. Thus, M_g *does* retract onto the non separating circle C', as desired. ☐

(14) Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0 cells, four 1 cells, three 2 cells, and one 3 cell. Using this structure, show that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i \pm j, \pm k\}$ of order eight.

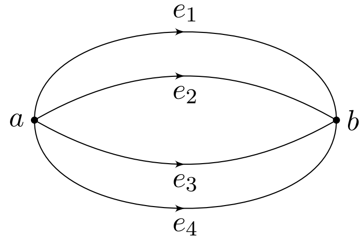
Proof. The cube I^3 can be seen as a cell complex with eight 0 cells (vertices), twelve 1 cells (edges), six 2 cells, and one 3 cell. The identifications described by the right-handed screw motion leave the 3 cell unchanged and merge opposite pairs of 2 cells (faces). Hence, there is one 3 cell and $6/2 = 3$ total 2 cells in the quotient. The identifications of the 0 and 1 cells in the quotient space are depicted below. Let i, j, k be group elements whose action is translation by one unit in the x, y, or z directions. Assume the directions are such that $+x$ is facing the reader, $+y$ is rightward, and $+z$ is straight up.



For example, we see that the horizontal edge e_2 (in blue) in the frontmost face is sent, when acted on by direction i, to an edge on the back of the cube pointing upward. When acted upon by i again, the edge is sent to the e_2 in the left vertical face. Finally, when i acts on this resultant edge, e_2 is sent back to its initial position. Similarly, by acting on 1 cells with j and k, we get orbits of size 3. Since every 1 cell is contained in an orbit of size 3 (and there are twelve one cells), the quotient will have $12/3 = 4$ distinct 1 cells (e_1, e_2, e_3, e_4). Furthermore, the screw motion identifies the 0 cells such that there are only two distinct 0 cells in the quotient; these cells have been labeled a and b. Thus, the quotient space X is a cell complex with two 0 cells, four 1 cells, three 2 cells, and one 3 cell.

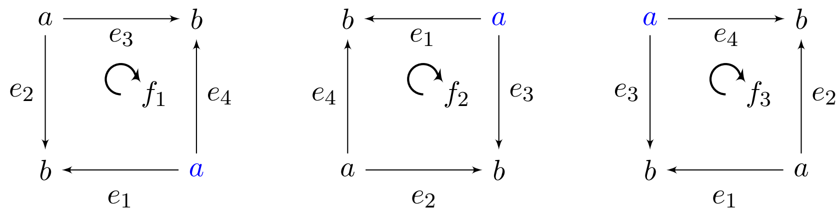
We now analyze $\pi_1(X)$. Since X is path-connected, the inclusion $X^2 \hookrightarrow X$ induces an isomorphism $\pi_1(X^2) \cong \pi_1(X)$ by Proposition 1.26 (c). So we examine the 2-skeleton $\pi_1(X^2)$. In order to do so, we determine the 1-skeleton X^1 and then attach the three 2-cells in the quotient X.

By the identifications drawn in the cube above, X^1 has the form depicted in the figure below. Then by the map that collapses e_1, we see that X^1 is homeomorphic to the wedge of three circles with generators $e_1 \cdot \overline{e_2}$, $e_1 \cdot \overline{e_3}$, and $e_1 \cdot \overline{e_4}$. So $\pi_1(X^1) \cong \pi_1(S^1 \vee S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ with generators $i := [e_1 \cdot \overline{e_2}], j := [e_1 \cdot \overline{e_3}], k := [e_1 \cdot \overline{e_4}]$.



Denote the three distinct 2 cells (faces) as f_1, f_2, f_3 where f_1 is the top horizontal face in the diagram, f_2 is the frontmost (vertical) face, and f_3 is the rightmost vertical face. With the righthand screw motion, f_1 has the boundary $e_1 \cdot \overline{e_2} \cdot e_3 \cdot \overline{e_4}$, as seen in the diagram below. Thus,

f_1 provides the following relation: $\mathrm{id}_{\pi_1(X^2)} = [e_1 \cdot \overline{e_2} \cdot e_3 \cdot \overline{e_4}]$. Note that the shared basepoint of each loop (according to the I^3 diagram above) is colored blue.



Similarly, f_2 and f_3 provide the following relations (noting that each loop begins at the same basepoint):

$$\begin{aligned}\mathrm{id}_{\pi_1(X^2)} &= [e_3 \cdot \overline{e_2} \cdot e_4 \cdot \overline{e_1}] \\ \mathrm{id}_{\pi_1(X^2)} &= [e_4 \cdot \overline{e_2} \cdot e_1 \cdot \overline{e_3}].\end{aligned}$$

Expressing these relations in terms of the generators,

$$\begin{aligned}\mathrm{id}_{\pi_1(X^2)} &= [e_1 \cdot \overline{e_2} \cdot e_3 \cdot \overline{e_4}] = [e_1 \cdot \overline{e_2}][e_3 \cdot (\overline{e_1} \cdot e_1) \cdot \overline{e_4}] = [e_1 \cdot \overline{e_2}][e_1 \cdot \overline{e_3}]^{-1}[e_1 \cdot \overline{e_4}] = \boxed{ij^{-1}k} \\ \mathrm{id}_{\pi_1(X^2)} &= [e_3 \cdot \overline{e_2} \cdot e_4 \cdot \overline{e_1}] = [e_3 \cdot (\overline{e_1} \cdot e_1) \cdot \overline{e_2} \cdot e_4 \cdot \overline{e_1}] = [e_1 \cdot \overline{e_3}]^{-1}[e_1 \cdot \overline{e_2}][e_1 \cdot \overline{e_4}]^{-1} = \boxed{j^{-1}ik^{-1}} \\ \mathrm{id}_{\pi_1(X^2)} &= [e_4 \cdot \overline{e_2} \cdot e_1 \cdot \overline{e_3}] = [e_4 \cdot (\overline{e_1} \cdot e_1) \cdot \overline{e_2} \cdot e_1 \cdot \overline{e_3}] = [e_1 \cdot \overline{e_4}]^{-1}[e_1 \cdot \overline{e_2}][e_1 \cdot \overline{e_3}] = \boxed{k^{-1}ij}\end{aligned}$$

Hence, since X^2 is obtained from X^1 by attaching the 2 cells f_1, f_2, f_3, by Proposition 1.2.6 (a) we have that $\pi_1(X^2) \cong \pi_1(X^1)/\langle ij^{-1}k, j^{-1}ik^{-1}, k^{-1}ij\rangle$. So

$$\begin{aligned}\pi_1(X) &\cong \pi_1(X^2) \cong (\mathbb{Z} * \mathbb{Z} * \mathbb{Z})/\langle ij^{-1}k, j^{-1}ik^{-1}, k^{-1}ij\rangle \\ &\cong (\mathbb{Z} * \mathbb{Z} * \mathbb{Z})/\langle ki = j, ij = k, i = jk\rangle.\end{aligned}$$

Now observe that

$$i^2 = i(jk) = (ij)k = k^2 = k(ij) = (ki)j = j^2$$

and

$$(i^2)^2 = i^2j^2 = (k^{-1}k)i^2j^2 = (k^{-1}k)i(ij)j = (k^{-1}k)i(k)j = k^{-1}(jk)j = k^{-1}ij = \mathrm{id}_{\pi_1(X^2)}.$$

So by defining $i^2 = j^2 = k^2 := -1$ and $\mathrm{id}_{\pi_1(X^2)} := 1$, we can produce the following multiplication table:

	1	**i**	**j**	**k**	**-1**	**-i**	**-j**	**-k**
1	1	i	j	k	-1	-i	-j	-k
i	i	-1	k	-j	-i	1	-k	j
j	j	-k	-1	i	-j	k	1	-i
k	k	j	-i	-1	-k	-j	i	1
-1	-1	-i	-j	-k	1	i	j	k
-i	-i	1	-k	j	i	-1	k	-j
-j	-j	k	1	-i	j	-k	-1	i
-k	-k	-j	i	1	k	j	-i	-1

For example, kj is computed by $kj = k(1)j = k(k^{-1}ij)j = ij^2 = i(-1) = -i$, and the other calculations are similar. Since this multiplication table is identical to that of the quaternions $\mathbb{Q}_8$, we conclude that $\pi_1(X)$ is the quaternion group $\{\pm 1, \pm i \pm j, \pm k\}$ of order eight. □