MATHEMATICS 215A: HOMEWORK 3

LAURA LYMAN

(1) Suppose that a simplicial complex X is covered by two subcomplexes Y and Z such that $H_*(Y)$ and $H_*(Z)$ are both zero in positive degrees and $\simeq \mathbb{Z}$ for $* = 0$. Can you compute $H_*(X)$ in terms of the homology of the complexes Y, Z, and $Y \cap Z$?

Proof. Yes. Since $X = Y \cup Z$, Meyer-Vietoris yields the long exact sequence

$$\begin{aligned} H_{n+1}(X) &\to H_n(Y \cap Z) \to H_n(Y) \oplus H_n(Z) \to H_n(X) \to H_{n-1}(Y \cap Z) \to H_{n-1}(Y) \oplus H_{n-1}(Z) \\ \cdots &\to H_1(Y) \oplus H_1(Z) \to H_1(X) \to H_0(Y \cap Z) \to H_0(Y) \oplus H_0(Z) \to H_0(X) \to 0. \end{aligned}$$

Since $H_n(Y) \oplus H_n(Z) \simeq 0$ for $n \geq 1$, we have

$$\begin{aligned} H_{n+1}(X) &\to H_n(Y \cap Z) \to 0 \to H_n(X) \to H_{n-1}(Y \cap Z) \to 0 \\ \cdots &\to H_1(Y \cap Z) \to 0 \to H_1(X) \to H_0(Y \cap Z) \to \cdots \end{aligned}$$

Then $\ker(H_{n-1}(Y \cap Z) \to 0) = H_{n-1}(Y \cap Z)$ and $\operatorname{Im}(0 \to H_n(X)) = 0$, so by exactness $H_n(X) \overset{\approx}{\to} H_{n-1}(Y \cap Z)$ is a bijection for $n \geq 2$. Thus

$$\boxed{H_n(X) \simeq H_{n-1}(Y \cap Z) \quad (n \geq 2).}$$

From here, we examine different possibilities for $Y \cap Z$. By Proposition 2.7, $H_0(S)$ for a topological space S is the direct sum of copies of $\mathbb{Z}$, one per path-connected component of S. Since $H_0(Y), H_0(Z) \simeq \mathbb{Z}$, Y and Z each have one path-connected component.

Suppose $Y \cap Z \neq \emptyset$. Since $X = Y \cup Z$ and $Y \cap Z \neq \emptyset$, X must only have one path-connected component. (If X had more than one path-connected component, then X and Y could not be on the same component for otherwise their union would not equal all of X. However, if X and Y were on separate components, then $X \cap Y = \emptyset$ and there is a contradiction.) So X has one path-connected component, meaning $H_0(X) \simeq \mathbb{Z}$. Then the end of the long exact sequence is

$$0 \to H_1(X) \to H_0(Y \cap Z) \overset{\varphi}{\to} \mathbb{Z} \oplus \mathbb{Z} \overset{\psi}{\to} \mathbb{Z} \to 0$$

since $H_0(Y) \oplus H_0(Z) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Note that $\psi(x, y) = x + y$ (as seen on the top of page 150). Then $\ker \psi = \{(x, -x) \mid x \in \mathbb{Z}\}$ and thus $\operatorname{Im}(\varphi) = \{(x, -x) \mid x \in \mathbb{Z}\} \simeq \mathbb{Z}$ by exactness. From here, we deduce that $\ker(\varphi) \simeq H_0(Y \cap Z)/\mathbb{Z}$. So by the First Isomorphism Theorem and exactness,

$$H_1(X) \simeq H_1(X)/0 \simeq \operatorname{Im}(H_1(X) \to H_0(Y \cap Z)) \simeq \ker(\varphi) \simeq H_0(Y \cap Z)/\mathbb{Z}.$$

Hence

$$\boxed{H_0(X) \simeq \mathbb{Z} \quad \text{and} \quad H_1(X) \simeq H_0(Y \cap Z)/\mathbb{Z} \quad (Y \cap Z \neq \emptyset).}$$

Now suppose $Y \cap Z = \emptyset$. Then $H_0(Y \cap Z) \simeq 0$ so the sequence becomes

$$\cdots \to 0 \to H_1(X) \to 0 \to \mathbb{Z} \oplus \mathbb{Z} \to H_0(X) \to 0.$$

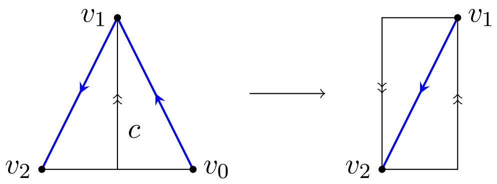
By exactness, $H_1(X) \to 0$ is an isomorphism and so $H_1(X) \simeq 0$. Furthermore, since $X = Y \cup Z$ and $Y \cap Z = \emptyset$, $X = Y \sqcup Z$ where Y and Z each have one path-connected component. So X has two path-connected components, meaning $H_0(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$. This also follows from the exact sequence; $\ker(\mathbb{Z} \oplus \mathbb{Z} \to H_0(X)) = \operatorname{Im}(0 \to \mathbb{Z} \oplus \mathbb{Z}) = 0$ and $\operatorname{Im}(\mathbb{Z} \oplus \mathbb{Z} \to H_0(X)) = \ker(H_0(X) \to 0) = H_0(X)$, so $\mathbb{Z} \oplus \mathbb{Z} \to H_0(X)$ is an isomorphism. Therefore

$$\boxed{H_0(X) \simeq \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad H_1(X) \simeq 0 \quad (Y \cap Z = \emptyset)}$$

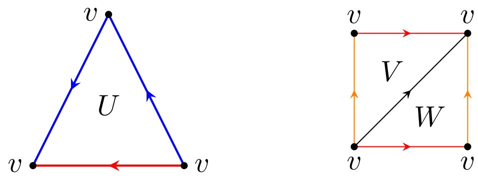
which completes the calculations. □

(2) Triangulate the space in problem (21) of HW 2 and compute its homology.

Proof. Let X be the space in question. By 2.1.1 of HW 2, the Möbius band can be triangulated by identifying two edges of a triangle in a way that preserves the order of the vertices, as pictured below.



By using this triangulation, and splitting the canonical gluing diagram of the torus into two triangles, we obtain a triangulation for the desired space.



Note that the red edge on the Möbius triangulation (boundary circle) is correctly glued to an $S^1 \times \{x_0\}$ factor of the torus (red edges) to produce X. Let e_1, e_2, e_3, e_4 denote the blue edges, red edges, orange edges, and black edge respectively. Then $C_0(X) = \langle v \rangle \simeq \mathbb{Z}$, $C_1(X) \simeq \langle e_1, e_2, e_3, e_4 \rangle \simeq \mathbb{Z}^4$, and $C_2(X) \simeq \langle U, V, W \rangle \simeq \mathbb{Z}^3$ (where U, V, W are the faces pictured above). Consider the chain diagram

$$0 \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0.$$

Using the definition of boundary maps, $\partial_1(e_1) = \partial_1(e_2) = \partial_1(e_3) = \partial_1(e_4) = v - v = 0$. Hence, ∂_1 is the zero map. Also, we compute that

$$\begin{aligned} \partial_2(U) &= e_1 + e_1 - e_2 = 2e_1 - e_2 \\ \partial_2(V) &= e_4 - e_2 - e_3 \\ \partial_2(W) &= e_2 + e_3 - e_4. \end{aligned}$$

Since $\partial_2(W) = -\partial_2(V)$, we have that $\text{Im}(\partial_2) = \langle 2e_1 - e_2, e_4 - e_2 - e_3 \rangle \simeq \mathbb{Z}^2$. Since $C_2(X) \simeq \mathbb{Z}^3$ and $C_2(X)/\ker \partial_2 \approx \text{Im}(\partial_2) \simeq \mathbb{Z}^2$ (First Isomorphism Theorem), we conclude that $\ker \partial_2 \simeq \mathbb{Z}$. So we compute that

$$\begin{aligned} H_0(X) &= \ker \partial_0 / \text{Im } \partial_1 = C_0(X)/0 \simeq \mathbb{Z} \\ H_1(X) &= \ker \partial_1 / \text{Im } \partial_2 = C_1(X)/\text{Im } \partial_2 \simeq \mathbb{Z}^4/\mathbb{Z}^2 \simeq \mathbb{Z}^2 \\ H_2(X) &= \ker \partial_2 / \text{Im } \partial_3 = \ker \partial_2 / 0 \simeq \mathbb{Z}. \end{aligned}$$

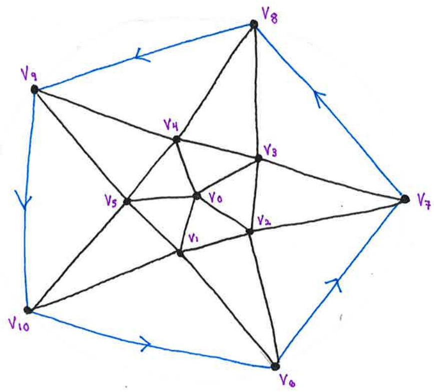
Hence,

$$\boxed{H_n(X) \simeq \begin{cases} \mathbb{Z} & \text{if } n = 0, 2 \\ \mathbb{Z}^2 & \text{if } n = 1 \\ 0 & \text{else.} \end{cases}}$$

Alternatively, the homology of X could have been computed with Meyer-Vietoris by picking neighborhoods of the boundary circle contained entirely in the torus and Möbius band respectively. □

(3) The *mod n Moore space* is defined as the quotient of the unit disc in $\mathbb{C}$ by the equivalence relation for which any ζ with $\zeta\overline{\zeta} = 1$ is identified with $\zeta\mu_n$, where μ_n is a primitive n-th root of unity. Find a triangulation of the modulo 5 Moore space and compute its homology.

Proof. Considering the CW structure, we have five 0-cells, five 1-cells, and one 2-cell, where the attaching map glues the 2-cell along the boundary of 1 cells by wrapping itself around all of them five times; that is, the attaching map has the form $z \mapsto z^5$. So the attaching map has degree 5, and all of the blue edges in the triangulation below are identified.



Consequently, the CW structure yields that

$$\boxed{H_i(X) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}_5 & \text{if } i = 1 \\ 0 & \text{else.} \end{cases}}$$

Alternatively, we could have used the triangulation pictured above. The middle vertex v_0 (call it a) is not identified with any vertices, while the remaining vertices (call them b) are all identified; this yields 2 vertices (a and b), 6 edges (one from b to b after the five outside blue edges are identified, and five between a and the b vertices $v_1, \dots, v_5$), and 5 faces (the five triangles in the inner pentagon). However, using the CW structure seems to be more efficient for computing homology. $\square$

(4) For any simplicial complex X, the n-skeleton $X^{(k)}$ of X is the simplicial complex with the same vertex set as X and whose collection of simplices consists of the simplices of X with cardinality up to (and including) $k+1$. Compute the homology of the 3-skeleton $X^{(3)}$ for $X = \Delta[5]$, the standard 5-simplex.

Proof. Since $X^{(3)}$ has no k-cells for $k \geq 4$, $H_k(X^{(3)}) = 0$ for all $k \geq 4$ (the boundary maps are forced to be zero maps in that part of the chain complex). Furthermore, since $X^{(3)}$ is non-empty and path-connected, $H_0(X^{(3)}) = \mathbb{Z}$. Now, the observation to make here is that $H_i(X^{(3)}) = H_i(X)$ for $i < 3$, since $X^{(3)}$ inherits its simplicial complex structure from X, meaning the chain complex for $X^{(3)}$ has the same groups/maps as X does at its end for $0 \leq i \leq 2$. Since $X = \Delta[5]$ is contractible, $H_0(X) = \mathbb{Z}$ and $H_i(X) = 0$ for $i \neq 5$. Hence, $H_1(X^{(3)}) = H_2(X^{(3)}) = 0$.

To compute $H_3(X^{(3)})$, we can proceed by using Theorem 2.44 or by explicitly determining the ∂_3 boundary map. For the latter method, we know that $H_3(X^{(3)}) = \ker(\partial_3)/\{0\} = \ker(\partial_3)$ and there are $\binom{5+1}{3} = 20$ total 2-simplices. Label the vertices $0, \dots, 5$ with standard orientation and let *abcd* denote the boundary of the tetrahedron on vertices a, b, c, d. Then we can compute (for example) $\partial_3(0124) = 124 - 024 + 014 - 012$. With similar calculations, ∂_3 can be represented as the following matrix

012	013	014	015	023	024	025	034	035	045	123	124	125	134	135	145	234	235	245	345
−1	1			−1						1									
−1		1			−1						1								
−1			1			−1						1							
	−1	1					−1						1						
	−1		1					−1						1					
		−1	1						−1						1				
				−1	1		−1									1			
				−1		1		−1									1		
					−1	1			−1									1	
							−1	1	−1										1
										−1	1		−1			1			
										−1		1		−1			1		
											−1	1			−1			1	
													−1	1	−1				1
																−1	1	−1	1

which has a rank of 15; therefore, $\ker(\partial_3) \simeq \mathbb{Z}^5$ and so $H_3(X^{(3)}) \simeq \mathbb{Z}^5$. We can verify this calculation with Theorem 2.44, which states that

$$\chi(X^{(3)}) = \text{rank}(H_0(X^{(3)})) - \text{rank}(H_1(X^{(3)})) + \text{rank}(H_2(X^{(3)})) - \text{rank}(H_3(X^{(3)})) + 0$$

where χ denotes the Euler characteristic (and note that the *rank* of a finitely generated abelian group is the number of $\mathbb{Z}$ summands). Since $X^{(3)}$ has $\binom{6}{4} = 15$ 3-cells, $\binom{6}{3} = 20$ 2-cells, $\binom{6}{2} = 15$ 1-cells, and 6 0-cells, $\chi(X^{(3)}) = 6 - 15 + 20 - 15 = -4$. Hence

$$-4 = 1 - 0 + 0 - \text{rank}(H_3(X^{(3)})) + 0$$

and so $H_3(X^{(3)}) \simeq \mathbb{Z}^5$. Therefore, we conclude that

$$\boxed{H_i(X^{(3)}) \simeq \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}^5 & \text{if } i = 3 \\ 0 & \text{else} \end{cases}}$$

which completes the homology calculations. □

(5) Let $\mathcal{P}$ denote a finite partially ordered set. Let the *nerve* of P, called $\mathcal{N}(P)$, be the simplicial complex whose vertex set is the underlying set of $\mathcal{P}$ and whose k-simplices are the sets $\{p_0, \ldots, p_k\}$ for which the restriction of the partial order on $\mathcal{P}$ is a total ordering. Let $\mathcal{P}_n$ denote the partially ordered set consisting of all proper non-trivial subsets of $\{0, 1, \ldots, n+1\}$. Compute the homology of the nerves of $\mathcal{P}_1$ and $\mathcal{P}_2$. Draw the geometry realizations and make a guess about the homology of the nerve $\mathcal{P}_n$ for all $n \in \mathbb{N}$.

Proof. Geometrically, $\mathcal{P}_1$ is realized as the boundary of a triangle; that is, $\mathcal{P}_1$ is the 2-simplex with its one face $[v_0, v_1, v_2]$ removed. Similarly, $\mathcal{P}_2$ is realized as the boundary of a tetrahedron, meaning $\mathcal{P}_2$ is the 3-simplex except that its interior is not filled in. Continuing in this matter, we conjecture that $\mathcal{P}_n$ is the $(n+1)$-simplex without its face of highest dimension; equivalently, $\mathcal{P}_n$ is the boundary of the standard $(n+1)$-simplex. Thus, $\mathcal{P}_n$ is homeomorphic to S^n and therefore $H_i(\mathcal{P}_n) \simeq \mathbb{Z}$ for $i = 0, n$ and is trivial otherwise. Similarly, $N(\mathcal{P}_1)$ and $N(\mathcal{P}_2)$ are barycentric subdivisions of the simplicial structures on S^1 and S^2 induced by their respective homeomorphisms to Δ^2 and Δ^3 (without their top-dimensional simplices). Hence,

$$H_i(N(\mathcal{P}_1)) = H_i(S^1) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 1 \\ 0 & \text{else} \end{cases} \qquad H_i(N(\mathcal{P}_2)) = H_i(S^2) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2 \\ 0 & \text{else.} \end{cases}$$

Therefore, we hypothesize that $H_i(N(\mathcal{P}_n)) \simeq H_i(S^n) = \mathbb{Z}$ for $i = 0, n$ and $H_i(N(\mathcal{P}_n)) = 0$ otherwise. □

(6) Let $\mathcal{P}$ be a partially ordered set. Let $\mathcal{P}^{\text{op}}$ be the same partially ordered set with opposite partial ordering; that is, $x \leq_{\mathcal{P}} x' \Leftrightarrow x' \leq_{\text{op}} x$. Show that there is a canonical isomorphism of simplicial complexes $N(\mathcal{P}) \simeq N(\mathcal{P}^{\text{op}})$. Conclude that the map of partially ordered sets

$$\mathcal{P}_n \longrightarrow \mathcal{P}_n^{\text{op}} \quad S \longmapsto \{0, \ldots, n+1\} \setminus S$$

induces a group action of $\mathbb{Z}/2\mathbb{Z}$ on $N(\mathcal{P}_n)$. For $n = 1, 2$, evaluate the effect of this map on homology.

Proof. Consider the isomorphism

$$N(\mathcal{P}) \longrightarrow N(\mathcal{P}^{\text{op}}) \quad [v_0, \ldots, v_k] \longmapsto [v_k, \ldots, v_0]$$

meaning the isomorphism reverses the order of all partial inequalities; that is, the k-simplex $v_0 \leq \cdots \leq v_k$ in $N(\mathcal{P})$ is sent to a k-simplex in $N(\mathcal{P}^{\text{op}})$ with partial ordering $v_k \leq \cdots \leq v_0$. Furthermore, note that each restriction of this map to a face preserves the vertex ordering (ignoring vertices preserves the ordering of the remaining vertices).

We say a map $f : S \to T$ between posets S, T is *order-preserving* if $\forall\ x, y \in S$, $x \leq y \Rightarrow f(x) \leq f(y)$. Furthermore, a map $f : S \to T$ is *order-reflecting* if $\forall\ x, y \in S$, $f(x) \leq f(y) \Rightarrow x \leq y$. A map that is both order-preserving and order-reflecting is said to be an *order-embedding*. Finally, partially ordered sets P_n and P_n^{op} are *isomorphic* if there exists a bijective order-embedding $P_n \to P_n^{\mathrm{op}}$. Now consider

$$\mathcal{P}_n \longrightarrow \mathcal{P}_n^{\mathrm{op}} \quad S \longmapsto \{0, \dots, n+1\} \setminus S$$

and let f denote this map. For $S, T \in \mathcal{P}_n$

$$\begin{aligned} S \leq_{\mathcal{P}} T \Leftrightarrow S \subseteq T & \\ \Leftrightarrow \{0, \dots, n+1\} \setminus S \supseteq \{0, \dots, n+1\} \setminus T & \quad [\text{since } S, T \subset \{0, \dots, n+1\}] \\ \Leftrightarrow f(S) \supseteq f(T) & \quad [\text{by definition of } f] \\ \Leftrightarrow f(S) \geq_{\mathcal{P}} f(T) \Leftrightarrow f(S) \leq_{\mathrm{op}} f(T) & \quad [\text{by definition of } \mathcal{P}^{\mathrm{op}}.] \end{aligned}$$

Hence, f is order-preserving and order-reflecting; therefore f is an order-embedding. Since f is order-reflecting, f is injective ($f(S) \leq_{\mathrm{op}} f(T) \Rightarrow S \leq_{\mathcal{P}} T$ and so $f(S) = f(T) \Rightarrow S = T$). Now let $Y \in \mathcal{P}_n^{\mathrm{op}}$ be arbitrary (and so $Y \subset \{0, \dots, n+1\}$). Then $\{0, \dots, n+1\} \setminus Y \in \mathcal{P}_n$ and $f(\{0, \dots, n+1\} \setminus Y) = Y$. Hence, f is surjective and therefore is a bijection. Since f is a bijective order-embedding between $\mathcal{P}_n$ and P_n^{op}, we conclude that $\mathcal{P}_n \approx P_n^{\mathrm{op}}$.

Thus, we have an induced map between $N(\mathcal{P}_n)$ and $N(\mathcal{P}_n^{\mathrm{op}})$ (in fact, we probably only needed f to be order-preserving to yield an induced map). By composing with the "reversing isomorphism" described at the start, we have a map $g : N(\mathcal{P}_n) \to N(\mathcal{P}_n)$. Finally, since the induced map $N(\mathcal{P}_n) \to N(\mathcal{P}_n^{\mathrm{op}})$ and the isomorphism $N(\mathcal{P}) \to N(\mathcal{P}^{\mathrm{op}})$ are both involutions, the map $g : N(\mathcal{P}_n) \to N(\mathcal{P}_n)$ is an involution as well. So $\langle g \rangle = \{\mathbb{1}, g\} \approx \mathbb{Z}/2\mathbb{Z}$, and therefore $\mathbb{Z}/2\mathbb{Z}$ acts on $N(\mathcal{P}_n)$ as desired. Since g can be associated with a symmetry around a sphere, we have that g induces $\mathbb{1} : H_1(|N(\mathcal{P}_1|) \to H_1(|N(\mathcal{P}_1|)$ and $-\mathbb{1} : H_2(|N(\mathcal{P}_2|)) \to H_2(|N(\mathcal{P}_2|))$; therefore, we have evaluated the effect on homology for $n = 1, 2$.

□