MATHEMATICS 215A: HOMEWORK 4

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(2) Show that the maps $G \xrightarrow{n} G$ and $H \xrightarrow{n} H$ multiplying each element by $n \in \mathbb{Z}$ induce multiplication by n in Ext(H,G).

Proof. Consider the free resolution constructed in Hatcher § 3.1,

$$0 \to F_1 \to F_0 \to H.$$

From here we dualize and use the maps induced by n multiplication to map the dual resolution to itself. That is, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(H,G) & \longrightarrow & \mathrm{Hom}(F_0,G) & \longrightarrow & \mathrm{Hom}(F_1,G) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \times\overline{n}} & & \downarrow{\scriptstyle \times\overline{n}} & & \downarrow{\scriptstyle \times\overline{n}} & & \\ 0 & \longrightarrow & \mathrm{Hom}(H,G) & \longrightarrow & \mathrm{Hom}(F_0,G) & \longrightarrow & \mathrm{Hom}(F_1,G) & \longrightarrow & 0 \end{array}$$

where $\times n : G \to G$ is multiplication by n and $\times\overline{n} = (\times n)\circ f(x)$ for $f(x)$ in $\mathrm{Hom}(H,-), \mathrm{Hom}(F_0,-)$, and $\mathrm{Hom}(F_1,G)$ respectively; that is, $\times\overline{n}$ is the image of $\times n$ in these homomorphism groups. Note that $\times\overline{n}$ is a chain map, because $\times\overline{n}(f(x)) = nf(x)$. Since $\times\overline{n}$ is a chain map, $\times\overline{n}$ induces multiplication by n on $\mathrm{Ext}(H,G)$ (since $\mathrm{Ext}(H,G)$ is the homology of the complex).

However, by Lemma 3.1 in Hatcher, $\times n : H \to H$ extends to the chain map

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \times\overline{n}} & & \\ 0 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

and is unique up to chain homotopy. Note that $\alpha : F_1 \to F_1$ and $\beta : F_0 \to F_0$ are both multiplication by n to ensure that the diagram commutes. Now, let $\overline{\alpha}$ and $\overline{\beta}$ be the images of α and β in $\mathrm{Hom}(-,G)$. Then applying $\mathrm{Hom}(-,G)$ yields

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(H,G) & \longrightarrow & \mathrm{Hom}(F_0,G) & \longrightarrow & \mathrm{Hom}(F_1,G) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \times\overline{n}} & & \uparrow{\scriptstyle \overline{\beta}} & & \uparrow{\scriptstyle \overline{\alpha}} & & \\ 0 & \longrightarrow & \mathrm{Hom}(H,G) & \longrightarrow & \mathrm{Hom}(F_0,G) & \longrightarrow & \mathrm{Hom}(F_1,G) & \longrightarrow & 0. \end{array}$$

Since $\overline{\alpha}, \overline{\beta}$, and $\times\overline{n}$ are all multiplication by n, the map induced on homology is also multiplication by n; therefore, $\mathrm{Ext}(H,G)$ is multiplication by n, as desired. □

(3) Regarding $\mathbb{Z}_2$ as a module over the ring $\mathbb{Z}_4$, construct a resolution of $\mathbb{Z}_2$ by free modules over $\mathbb{Z}_4$ and use this to show that $\mathrm{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2)$ is nonzero for all n.

Proof. Let $\mathbb{Z}_4 \to \mathbb{Z}_2$ denote the quotient map. Let $\mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4$ denote multiplication by 2. Consider the free resolution

$$\cdots \longrightarrow \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

Since $\mathrm{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \simeq \mathbb{Z}_2$ (by mapping the generator of $\mathbb{Z}_4$ to either 0 or 1), the dual of $\times 2 : \mathbb{Z}_4 \to \mathbb{Z}_4$ is simply the zero map. Hence, we have the dual sequence

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \longrightarrow \cdots .$$

such that the final $\mathbb{Z}_2 \xrightarrow{\mathbb{1}} \mathbb{Z}_4$ is the identity map. Then the homology of this complex is $\mathbb{Z}_2/\{0\} \approx \mathbb{Z}_2$ for *every* degree, meaning (since $\text{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2)$ is the homology of the complex) that $\text{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2)$ is nonzero for all $n \in \mathbb{N}$. □

(4) What happens if one defines homology groups $h_n(X; G)$ as the homology groups of the following chain complex?

$$\cdots \to \text{Hom}(G, C_n(X)) \to \text{Hom}(G, C_{n-1}(X)) \to \cdots$$

More specifically, what are the groups $h_n(X; G)$ when $G = \mathbb{Z}, \mathbb{Z}_m$, and $\mathbb{Q}$?

Proof. Suppose $G = \mathbb{Z}_m$. Then since $C_n(X)$ is free abelian (and therefore has no torsion elements), any $f : \mathbb{Z}_m \to C_n(X)$ is a trivial map. Hence, $\text{Hom}(G, C_n(X)) = 0$ for all $n \in \mathbb{N}$, so the dual complex satisfies

$$h_n(X; \mathbb{Z}_m) = 0 \qquad \text{for all } n \in \mathbb{N}.$$

Now suppose $G = \mathbb{Z}$, so $C_n(X)$ is a $\mathbb{Z}$-module. Since $\text{Hom}_R(R, M) \simeq M$ for all R-modules M (by the map $f \mapsto f(1)$), we have that $\text{Hom}(\mathbb{Z}, C_n(X)) \simeq C_n(X)$. Therefore, the cochain complex is *exactly the same as the original chain complex.* That is,

$$h_n(X; G) \simeq H_n(X; \mathbb{Z}) \qquad \text{for all } n \in \mathbb{N}.$$

Finally, suppose $G = \mathbb{Q}$. Then $C_n(X)$ is a free $\mathbb{Z}$-module, so $f(1) = 0$ for all $f \in \text{Hom}(\mathbb{Q}, C_n(X))$ since any nonzero element of $\mathbb{Z}$ can divide $1_{\mathbb{Q}}$. Thus, $f = 0$ for arbitrary $f \in \text{Hom}(\mathbb{Q}, C_n(X))$, meaning $\text{Hom}(\mathbb{Q}, C_n(X)) = 0$. Thus,

$$h_n(X; \mathbb{Q}) = 0 \qquad \text{for all } n \in \mathbb{N}$$

which completes the calculations. □

(5) Regarding a cochain $\varphi \in C^1(X; G)$ as a function from paths in X to G, show that if φ is a cocycle, then
   (a) $\varphi(f \cdot g) = \varphi(f) + \varphi(g)$,
   (b) φ takes the value 0 on constant paths,
   (c) $\varphi(f) = \varphi(g)$ if $f \simeq g$,
   (d) φ is a coboundary $\Leftrightarrow$ $\varphi(f)$ depends only on the endpoints of f for all f. [In particular, (a) and (c) give a map $H^1(X; G) \to \text{Hom}(\pi_1(X), G)$, which the universal coefficient theorem says is an isomorphism if X is path-connected.]

   Proof. (a) Let $v_0 = (0, 0), v_1 = (1, 0)$, and $v_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Define $\Delta^2 = [v_0, v_1, v_2]$ and $\sigma : \Delta^2 \to X$ such that $\sigma(x_1, x_2) = f(g(x_1))$. Letting $\sigma_1 = \sigma\big|_{[v_0,v_1]}, \sigma_2 = \sigma\big|_{[v_0,v_2]}$ and $\sigma_3 = \sigma\big|_{[v_2,v_1]}$, we have $\sigma_1 = f \cdot g, \sigma_2 = f$, and $\sigma_3 = g$. Then

$$0 = \delta\varphi(\sigma) = \varphi(\partial\sigma) = \varphi(\sigma_1) - \varphi(\sigma_2) + \varphi(\sigma_3)$$

   since $\varphi \in C^1(X; G)$ is a cocycle. Substituting into the expression above, we have $0 = -\varphi(f \cdot g) - \varphi(f) + \varphi(g)$, or equivalently $\varphi(f \cdot g) = \varphi(f) + \varphi(g)$.

   (b) Let f be a constant path. Then we know $f \cdot f = f$. By part(a), $\varphi(f \cdot f) = \varphi(f) + \varphi(f)$. However, since $f = f \cdot f$, we have $\varphi(f \cdot f) = \varphi(f)$ as well. So $\varphi(f) = 2\varphi(f)$, meaning $\varphi(f) = 0$.

   (c) Now suppose two paths f, g are homotopic; that is, there exists some homotopy $h : I \times I \to X$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. Then consider the 2-simplicies

$$\sigma_1 : \Delta^2 \to X \qquad \sigma_1 = h\big|_{[(0,0),(1,0),(1,1)]}$$
$$\sigma_2 : \Delta^2 \to X \qquad \sigma_2 = h\big|_{[(0,0),(0,1),(1,1)]}.$$

The we compute that $\sigma_1\big|_{[(1,0),(1,1)]} = f(1) = g(1)$ and $\sigma_1\big|_{[(0,0),(1,0)]} = f$. Similarly, $\sigma_2\big|_{[(0,0),(0,1)]} = f(0) = g(0)$ and $\sigma_2\big|_{[(0,1),(1,1)]} = g$. Since φ is a cocycle,

$$\begin{aligned} 0 &= \delta(\varphi(\sigma_1)) = \varphi(\partial\sigma_1) \\ &= \varphi(\sigma_1\big|_{[(1,0),(1,1)]}) - \varphi(\sigma_1\big|_{[(0,0),(1,1)]}) + \varphi(\sigma_1\big|_{[(0,0),(1,0)]}) \\ &= \varphi(f(1)) - \varphi(\sigma_1\big|_{[(0,0),(1,1)]}) + \varphi(f) && \text{[substituting from above]} \\ &= 0 - \varphi(\sigma\big|_{[(0,0),(1,1)]}) + \varphi(f) && \text{[by part (b)].} \end{aligned}$$

Thus, $\varphi(f) = \varphi(\sigma_1\big|_{[(0,0),(1,1)]})$. By a similar computation, $0 = \varphi(g) - \varphi(\sigma_2\big|_{[(0,0),(1,1)]}) + 0$ and so $\varphi(g) = \varphi(\sigma_2\big|_{[(0,0),(1,1)]})$. However, since $\sigma_1\big|_{[(0,0),(1,1)]} = h\big|_{[(0,0),(1,1)]}$ and $\sigma_2\big|_{[(0,0),(1,1)]} = h\big|_{[(0,0),(1,1)]}$, we have $\sigma_1\big|_{[(0,0),(1,1)]} = \sigma_2\big|_{[(0,0),(1,1)]}$ and hence $\varphi(\sigma_1\big|_{[(0,0),(1,1)]}) = \varphi(\sigma_2\big|_{[(0,0),(1,1)]})$. Thus, $\varphi(f) = \varphi(g)$.

(d) ($\Rightarrow$) Suppose φ is a coboundary. By definition, $\varphi = \delta\phi$ for some $\phi \in C^0(X;G)$. Then for f an arbitrary path,

$$\varphi(f) = \delta(\phi(f)) = \phi(\partial f) = \varphi(f(1)) - \varphi(f(0)).$$

Therefore, $\varphi(f)$ depends only on the endpoints $f(0), f(1)$ of f, as desired.
($\Leftarrow$) Now suppose $\varphi(f)$ depends only on the endpoints of path f. As suggested by the hint, decompose X into its path-connected components $X = \sqcup_\alpha X_\alpha$. We wish to show $\varphi = \delta\phi$ for some $\phi \in C^0(X;G)$ so that φ is a coboundary
Consider an arbitrary path-connected component X_α and fix $x_\alpha \in X_\alpha$. Since X_α is path connected, for every $z \in X_\alpha$ there exists a path $f_{x_\alpha,z}$ such that $f(0) = x_\alpha$ and $f(1) = z$. Then let $\phi : \sqcup_\alpha X_\alpha \to X$ such that $\phi(z) = \varphi(f_{x_\alpha,z})$, where x_α is the fixed point corresponding to the unique connected component X_α containing z. Furthermore, if there exist homotopic paths $f_{x_\alpha,z} \simeq g_{x_\alpha,z}$ between x_α and z, we fill have $\varphi(f_{x_\alpha,z}) = \varphi(g_{x_\alpha,z})$ since φ depends only on its endpoints. Hence, ϕ is well-defined.
We now show that $\delta\phi(s) = \varphi(s)$ for each path $s \in C_1(X;G)$. Since $s : I \to X$ and the image of s is connected, we have $s : I \to X_\alpha$ for a unique α. Compute that

$$(**) \qquad \delta(\phi(s)) = \phi(\partial s) = \phi(s(1)) - \phi(s(0)) = \varphi(f_{x_\alpha,s(1)}) - \varphi(f_{x_\alpha,s(0)}),$$

where $f_{x_\alpha,s(0)}$ and $f_{x_\alpha,s(1)}$ are paths from x_α to $s(0)$ and x_α to $s(1)$, respectively.

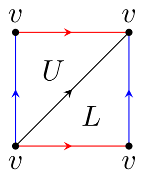
Let $f^{-1}_{x_\alpha,s(0)}$ be the inverse path of $f_{x_\alpha,s(0)}$; that is, $f^{-1}_{x_\alpha,s(0)}(t) = f_{x_\alpha,s(0)}(1-t)$. Since $f^{-1}_{x_\alpha,s(0)} \cdot f_{x_\alpha,s(0)}$ is a path from $s(0)$ to $s(0)$, and therefore homotopic to a constant map, $\varphi(f^{-1}_{x_\alpha,s(0)} \cdot f_{x_\alpha,s(0)}) = \varphi(s(0)) = 0$ by part (b) (where $s(0)$ denotes the constant path with value $s(0)$ in this context). Furthermore, $\varphi(f_{x_\alpha,s(0)} \cdot f^{-1}_{x_\alpha,s(0)}) = \varphi(f^{-1}_{x_\alpha,s(0)}) + \varphi(f_{x_\alpha,s(0)}) + \varphi(f^{-1}_{x_\alpha,s(0)})$ by part (a), meaning $0 = \varphi(f^{-1}_{x_\alpha,s(0)}) + \varphi(f_{x_\alpha,s(0)})$ and hence $\varphi(f^{-1}_{x_\alpha,s(0)}) = -\varphi(f_{x_\alpha,s(0)})$. Furthermore, note that $\varphi(f^{-1}_{x_\alpha,s(0)}) \cdot \varphi(f_{x_\alpha,s(1)})$ is a path from $s(0)$ to $s(1)$. Since φ only depends on its endpoints, we have $\varphi(f^{-1}_{x_\alpha,s(0)} \cdot f_{x_\alpha,s(1)}) = \varphi(s)$. Therefore

$$\begin{aligned} \varphi(s) &= \varphi(f^{-1}_{x_\alpha,s(0)} \cdot f_{x_\alpha,s(1)}) \\ &= \varphi(f^{-1}_{x_\alpha,s(0)}) + \varphi(f_{x_\alpha,s(1)}) && \text{[by part (a)]} \\ &= \varphi(f_{x_\alpha,s(1)}) - \varphi(f_{x_\alpha,s(0)}) \\ &= \delta(\phi(s)) && \text{[by equation (**)].} \end{aligned}$$

Hence, $\varphi(s) = \delta\phi(s)$ for $\phi \in C^0(X;G)$ and $s \in C_1(X;G)$ arbitrary. Thus, $\varphi = \delta\phi$, so φ is a coboundary. □

(6) Directly from the definitions, compute the simplicial cohomology groups of $S^1 \times S^1$ with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients, using the Δ-complex structure given in Section 2.1. Then do the same for $\mathbb{R}P^2$ and the Klein bottle.

Proof. First we compute the cohomology groups $H^i(S^1 \times S^1; \mathbb{Z})$. In §2.1, the torus $S^1 \times S^1$ has the following Δ-complex structure.



Let a denote the red edge, b denote the blue edge, and c be the black diagonal edge. Then $C_0(X;\mathbb{Z}) = \mathbb{Z}\langle v\rangle$, $C_1(X;\mathbb{Z}) = \mathbb{Z}\langle a,b,c\rangle$, and $\mathbb{C}_2(X;Z) = \mathbb{Z}\langle U,L\rangle$. So we have the chain diagram

$$0 \xrightarrow{\partial_3} C_2(X;\mathbb{Z}) \xrightarrow{\partial_2} C_1(X;\mathbb{Z}) \xrightarrow{\partial_1} C_0(X;\mathbb{Z}) \xrightarrow{\partial_0} 0$$

which is equivalent to

$$0 \xrightarrow{\partial_3} \mathbb{Z}\langle U,L\rangle \xrightarrow{\partial_2} \mathbb{Z}\langle a,b,c\rangle \xrightarrow{\partial_1} \mathbb{Z}\langle v\rangle \xrightarrow{\partial_0} 0.$$

For the edges, $\partial_1(a) = \partial_1(b) = \partial_1(c) = v - v = 0$. Thus, ∂_1 is the zero map. Furthermore,

$$\partial_2(U) = c - a - b \qquad \partial_2(L) = a + b - c$$

which can be represented as left multiplication by

$$\partial_2 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

To see why, observe that (when acting on basis elements U, L) we have

$$\partial_2(U) = \begin{pmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

which equals $c - a - b$ with respect to the basis of $\mathbb{Z}\langle a,b,c\rangle$. Similarly, $\partial_2(L) = a + b - c$ in $\mathbb{Z}\langle a,b,c\rangle$. So the chain complex becomes

$$0 \xrightarrow{0} \mathbb{Z}\langle U,L\rangle \xrightarrow{\begin{pmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}} \mathbb{Z}\langle a,b,c\rangle \xrightarrow{0} \mathbb{Z}\langle v\rangle \xrightarrow{0} 0.$$

The dual chain is then

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}^T} \mathbb{Z}^2 \to 0.$$

Note that $\begin{pmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}^T$ is rank 1, so its kernel is isomorphic to $\mathbb{Z}^2$. Thus, we compute that

$$H^0(S^1 \times S^1; \mathbb{Z}) \simeq \mathbb{Z}/\{0\} \simeq \mathbb{Z} \quad H^1(S^1 \times S^1; \mathbb{Z}) \simeq \mathbb{Z}^2/\{0\} \simeq \mathbb{Z}^2 \quad H^2(S^1 \times S^1; \mathbb{Z}) \simeq \mathbb{Z}^2/\mathbb{Z} \simeq \mathbb{Z}.$$

Therefore,

$$\boxed{H^i(S^1 \times S^1; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2 \\ \mathbb{Z}^2 & \text{if } i = 1 \\ 0 & \text{else.} \end{cases}}$$

We now compute $H^i(S^1 \times S^1; \mathbb{Z}_2)$. The cochain complex is

$$0 \xrightarrow{\partial_3} \mathbb{Z}_2\langle U,L\rangle \xrightarrow{\partial_2} \mathbb{Z}_2\langle a,b,c\rangle \xrightarrow{\partial_1} \mathbb{Z}_2\langle v\rangle \xrightarrow{\partial_0} 0.$$

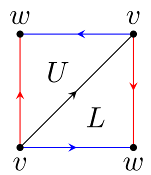
By dualizing, we have the same cochain as before, except that $\mathbb{Z}$ replaced with $\mathbb{Z}_2$ and $1 = -1$ in all maps. That is,

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix}1 & 1\\ 1 & 1\\ 1 & 1\end{pmatrix}^T} \mathbb{Z}^2 \to 0.$$

Therefore, $H^0(S^1 \times S^1; \mathbb{Z}_2) \simeq \mathbb{Z}_2/\{0\} \simeq \mathbb{Z}_2$, $H^1(S^1 \times S^1; \mathbb{Z}_2) \simeq \mathbb{Z}_2^2/\{0\} = \mathbb{Z}_2^2$, and $H^2(S^1 \times S^1; \mathbb{Z}_2) \simeq \mathbb{Z}_2^2/\mathbb{Z}_2 \simeq \mathbb{Z}_2$. Hence,

$$\boxed{H^i(S^1 \times S^1; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } i = 0, 2 \\ \mathbb{Z}_2^2 & \text{if } i = 1 \\ 0 & \text{else.} \end{cases}}$$

(b) Now we compute $H^i(\mathbb{R}P^2; \mathbb{Z})$ and $H^i(\mathbb{R}P^2; \mathbb{Z}_2)$. For the former, we have the following gluing diagram.



Again, let a be the red edge, b be the blue edge, and c be the black edge. The chain complex is

$$0 \xrightarrow{0} \mathbb{Z}\langle U, L\rangle \xrightarrow{\partial_2} \mathbb{Z}\langle a, b, c\rangle \xrightarrow{\partial_1} \mathbb{Z}\langle v, w\rangle \xrightarrow{0} 0.$$

Compute that

$$\partial_2(U) = c + b - a \qquad \partial_2(L) = c + a - b.$$
$$\partial_1(a) = w - v \qquad \partial_1(b) = w - v \qquad \partial_1(c) = v - v = 0.$$

Then we have

$$0 \xrightarrow{0} \mathbb{Z}\langle U, L\rangle \xrightarrow{\begin{pmatrix}-1 & 1\\ 1 & -1\\ 1 & 1\end{pmatrix}} \mathbb{Z}\langle a, b, c\rangle \xrightarrow{\begin{pmatrix}-1 & -1 & 0\\ 1 & 1 & 0\end{pmatrix}} \mathbb{Z}\langle v, w\rangle \xrightarrow{0} 0.$$

Dualizing yields

$$0 \to \mathbb{Z}^2 \xrightarrow{\begin{pmatrix}-1 & -1 & 0\\ 1 & 1 & 0\end{pmatrix}^T} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix}-1 & 1\\ 1 & -1\\ 1 & 1\end{pmatrix}^T} \mathbb{Z}^2 \to 0.$$

Now, the column space of $\partial_2^* = \begin{pmatrix}-1 & 1 & 1\\ 1 & -1 & 1\end{pmatrix}$ is $\mathrm{Im}(\partial_2^*)$ and is invariant under column recombinations

$$\begin{pmatrix}-1 & 1 & 1\\ 1 & -1 & 1\end{pmatrix} \to \begin{pmatrix}-1 & 0 & 0\\ 1 & 0 & 2\end{pmatrix}$$

and thus can be represented by the matrix $C = \begin{pmatrix}-1 & 0\\ 0 & 2\end{pmatrix}$. Since $|\det(C)| = 2$, the column space of ∂_2^* is a subgroup of index 2 in $\mathbb{Z}^2$. Since subgroups of index 2 are always normal, and a normal subgroup N of G satisfies $[G : N] = |G/N|$, we have that $|\mathbb{Z}^2/\mathrm{Im}(\partial_2 *)| = 2$. Thus, $H^2(\mathbb{R}P^2; \mathbb{Z})$ is a group of order 2, so $H^2(\mathbb{R}P^2; \mathbb{Z}) \simeq \mathbb{Z}_2$. Continuing these calculations, we have

$$\boxed{H^i(\mathbb{R}P^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}_2 & \text{if } i = 2 \\ 0 & \text{else.} \end{cases}}$$

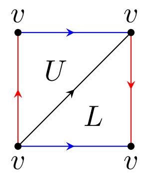
For $\mathbb{Z}_2$ coefficients, we have the same cochain complex with $\mathbb{Z}$ replaced by $\mathbb{Z}_2$ and $1 = -1$ when mapping into products of $\mathbb{Z}_2$. Thus, we have

$$0 \to \mathbb{Z}_2^2 \xrightarrow{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}^T} \mathbb{Z}_2^3 \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}^T} \mathbb{Z}_2^2 \to 0$$

which gives

$$\boxed{H^i(\mathbb{R}P^2; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } i = 0, 1, 2 \\ 0 & \text{else.} \end{cases}}$$

Finally, we compute the cohomology groups of the Klein bottle K with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients. Given the identification instructions from the following diagram, let a be the red edge, b be the blue edge, and c be the black edge.



Then the simplicial chain complex is

$$0 \xrightarrow{0} \mathbb{Z}\langle U, L \rangle \xrightarrow{\partial_2} \mathbb{Z}\langle a, b, c \rangle \xrightarrow{0} \mathbb{Z}\langle v \rangle \xrightarrow{0} 0.$$

Then

$$\partial_2(U) = c - b - a \qquad \partial_2(L) = c + a - b$$

and so we have

$$0 \xrightarrow{0} \mathbb{Z}\langle U, L \rangle \xrightarrow{\begin{pmatrix} -1 & 1 \\ -1 & -1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z}\langle a, b, c \rangle \xrightarrow{0} \mathbb{Z}\langle v \rangle \xrightarrow{0} 0.$$

Taking the dual,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} -1 & 1 \\ -1 & -1 \\ 1 & 1 \end{pmatrix}^T} \mathbb{Z}^2 \to 0.$$

Hence, we compute that

$$\boxed{H^i(K; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 1 \\ \mathbb{Z}_2 & \text{if } i = 2 \\ 0 & \text{else.} \end{cases}}$$

Similarly, with $\mathbb{Z}_2$ coefficients the cochain complex is

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2^3 \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}^T} \mathbb{Z}_2^2 \to 0,$$

and so

$$\boxed{H^i(K; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } i = 0, 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } i = 1 \\ 0 & \text{else.} \end{cases}}$$

This completes the desired calculations. □

(8) (a) Compute $H^i(S^n; G)$ by induction on n in two ways: using the long exact sequence of a pair and using the Mayer-Vietoris sequence.

Proof. Consider the pair (D^n, S^{n-1}). Then $D^n/S^{n-1} \simeq S^n$ has the cohomology sequence

$$\cdots \to H^i(D^n, S^{n-1}; G) \xrightarrow{j^*} H^i(D^n; G) \xrightarrow{\iota^*} H^n(S^{n-1}; G) \xrightarrow{\delta} H^{n+1}(D^n, S^{n-1}; G) \to \cdots$$

We wish to show that

$$H^i(S^n; G) = \begin{cases} G & \text{if } i = 0, n \\ 0 & \text{else} \end{cases}$$

which is consistent with Poincare duality. First note that when $i = 0$, $H_0(S^n; G) = G$ by connectedness, so $H^0(S^n; G) = \text{Hom}(H_0(S^n; G), G) = \text{Hom}(G, G) = G$. This holds for any $n \in \mathbb{N}$. Therefore, we focus on parts of the sequence with $i > 0$ from here.
Since D^n is contractible, $\widetilde{H}_i(D^n; G) = 0$ and so $\widetilde{H^i}(D^n; G) = \text{Hom}(0, G) = 0$. Hence, for $i > 0$ we have portions of the sequence as

$$\cdots \to 0 \to H^i(S^{n-1}; G) \to H^{i+1}(D^n, S^{n-1}; G) \to 0 \to \cdots$$

By exactness, there are isomorphisms $H^i(S^{n-1}; G) \cong H^{i+1}(D^n, S^{n-1}; G)$ for $i \geq 1$.

We now proceed by induction on n. For $n = 0$, we already showed that $H^0(S^0; G) = H^n(S^0; G) = G$. Furthermore, we know in general that $H^k(S^n; G) = 0$ for all $k > n$. This completes the base case.

For the inductive step, assume $H^i(S^{n-1}; G) = G$ for $i = 0, n-1$ and is 0 otherwise. Plugging $i = n-1$ into the isomorphism $H^i(S^{n-1}; G) \cong H^{i+1}(D^n, S^{n-1}; G)$, we have $H^{n-1}(S^{n-1}; G) = H^n(D^n, Sn-1; G)$. By the inductive hypothesis, we then have that $H^n(D^n, S^{n-1}; G) = G$. Since $D^n/S^{n-1} \simeq S^n$, by identifying the cohomology of the pair with the cohomology of the quotient, $H^n(S^n; G) = H^n(D^n, Sn-1; G) = G$. Similarly, $H^i(S^n; G) = 0$ for all $i \neq 0, n$. This completes the inductive step. Therefore,

$$H^i(S^n; G) = \begin{cases} G & \text{if } i = 0, n \\ 0 & \text{else.} \end{cases}$$

Now we complete the same calculation via Mayer Vietoris. Let $D_+ = \{v \in S^n \mid v_0 \geq 0\}$ and $D_- = \{v \in S^n \mid v_0 \leq 0\}$, so D_+ and D_- can be thought of as the northern/southern hemispheres of S^n. Let U, V be open neighborhoods of D_+ and D_- (respectively) overlapping at the equator (i.e. $\{v \in S^n \mid v_0 = 0\}$). Then $U \cap V$ is homeomorphic to S^{n-1} and $U \cup V = S^n$. By Mayer Vietoris, we have the cohomology sequence:

$$\cdots \to H^i(S^n; G) \xrightarrow{\psi} H^i(D_+; G) \oplus H^i(D_-; G) \xrightarrow{\phi} H^i(S^{n-1}; G) \to H^{i+1}(S^n; G) \to \cdots$$

Again, when $i = 0$, $H_0(S^n; G) = G$ by connectedness, so $H^0(S^n; G) = G$. This holds for any $n \in \mathbb{N}$. Thus, we focus on parts of the sequence with $i > 0$ from here. For $i > 0$, $H^i(D_+; G) = H^i(D_-; G) = 0$ by contractibility. Substituting into Mayer Vietoris yields

$$\cdots 0 \to H^i(S^{n-1}; G) \to H^{i+1}(S^n; G) \to 0 \to \cdots$$

So again, we have isomorphisms $H^i(S^{n-1}; G) \approx H^{i+1}(S^n; G)$ for $i \geq 0$.

Therefore, we are ready to induct over n. For $n = 0$, we already showed that $H^0(S^0; G) = H^n(S^0; G) = G$. Also, since generally $H^k(S^n; G) = 0$ for all $k > n, H^i(S^0; G) = 0$ for all $i \neq 0, n$. This completes the base case.

For the inductive step, assume $H^i(S^{n-1}; G) = G$ for $i = 0, n-1$ and is trivial otherwise. Since $H^i(S^{n-1}; G) \approx H^{i+1}(S^n; G)$, $H^i(S^n; G)$ corresponds to the correct cohomology groups for all $i \in \mathbb{N}$ as well. This completes the inductive step. □

(b) Show that if A is a closed subspace of X that is a deformation retract of some neighborhood, then the quotient map $X \to X/A$ induces isomorphisms $H^n(X, A; G) \approx \widetilde{H^n}(X/A; G)$ for all n.
Proof. Let V be a neighborhood of A that deformation retracts onto A. Consider the commutative diagram

$$\begin{array}{ccccc} H^n(X,A;G) & \longleftarrow & H^n(X,V;G) & \xrightarrow{\iota^*} & H^n(X-A,V-A;G) \\ \uparrow q^* & & \uparrow q^* & & \uparrow q^* \\ H^n(X/A,A/A;G) & \longleftarrow & H^n(X/A,V/A;G) & \xrightarrow{\iota^*} & H^n(X/A-A/A,V/A-A/A;G) \end{array}$$

where q^* is induced by the quotient map $q : X \to X/A$. By excision, the maps $H^n(X,V;G) \xrightarrow{\iota^*} H^n(X-A,V-A;G)$ and $H^n(X/A,V/A;G) \xrightarrow{\iota^*} H^n(X/A-A/A,V/A-A/A;G)$ are isomorphisms.

In the long exact sequence in homology of the triple (X,V,A), the groups $H_n(V,A) = 0$ for all $n \in \mathbb{N}$, since a deformation retraction of V onto A gives a homotopy equivalence of pairs $(V,A) \simeq (A,A)$, and $H_n(A,A) = 0$. Hence, the groups $H^n(V,A) = 0$ as well in the long exact sequence in cohomology of (X,V,A), so $H^n(X,V;G) \xrightarrow{\approx} H^n(X,A;G)$ is an isomorphism for each n. Since the deformation retraction of V onto A induces a deformation retraction of V/A onto A/A, the same argument proves that $H^n(X/A,V/A;G) \to H^n(X/A,A/A;G)$ is an isomorphism as well.

The right-hand map $q^*H^n(X/A-A/A,V/A-A/A;G) \to H^n(X-A,V-A;G)$ is an isomorphism, because q restricts to a homeomorphism on the complement of A. From the commutativity of the diagram, $q^* : H^n(X/A,V/A;G) \to H^n(X,V;G)$ is an isomorphism. By commutativity again, we have $q^* : H^n(X/A,A/A;G) \to H^n(X,A;G)$ is an isomorphism. Since $H^n(X/A,A/A;G) \approx \widetilde{H^n}(X/A;G)$, we have $H^n(X,A;G) \approx \widetilde{H^n}(X/A;G)$ for all $n \in \mathbb{N}$, as desired. □

(c) Show that if A is a retract of X then $H^n(X;G) \approx H^n(A;G) \oplus H^n(X,A;G)$.

Proof. Since A is a retract of X, there exists a continuous $r : X \to A$ such that $r|_A = \text{id}$, or equivalently $r \circ \iota = \text{id}_A$ for the inclusion $\iota : A \hookrightarrow X$. Since ι^* is injective, we have the following short exact sequence in cohomology:

$$0 \to H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{\iota^*} H^n(A;G) \to 0.$$

Since $r \circ \iota = \text{id}_A$, we have $\text{id} = \iota^* \circ r^* : H^n(A;G) \to H^n(A;G)$. By the Splitting Lemma, $H^n(X;G) \simeq H^n(A;G) \oplus H^n(X,A;G)$, completing the proof. □

(10) For the lens space $L_m(\ell_1,\dots,\ell_n)$ defined in Example 2.43, compute the cohomology groups using the cellular cochain complex and taking coefficients in $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_m$, and $\mathbb{Z}_p$ for p prime. Verify that the answers agree with those given by the universal coefficient theorem.

Proof. By Example 2.43, $L_m(\ell_1,\dots,\ell_n)$ has one cell e^k for each $0 \le k \le 2n-1$. So we have the cellular chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

when $G = \mathbb{Z}$. Dualizing yields

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

since $\text{Hom}(\mathbb{Z},\mathbb{Z}) = \mathbb{Z}$. When i is even and $2 \le i \le 2n-2$, we see that $H^i(L_m(\ell_1,\dots,\ell_n)) = \ker(\mathbb{Z} \to 0)/\text{Im}(\mathbb{Z} \xrightarrow{m} \mathbb{Z}) \simeq \mathbb{Z}/m\mathbb{Z} \simeq \mathbb{Z}_m$. When i is odd and $1 \le i < 2n-1$, $H^i(L_m(\ell_1,\dots,\ell_n)) = \ker(\mathbb{Z} \xrightarrow{m} \mathbb{Z})/\{0\} = \{0\}/\{0\} = 0$. Finally, $H^{2n-1}(L_m(\ell_1,\dots,\ell_n)) = \ker(\mathbb{Z} \to 0)/\text{Im}(\mathbb{Z} \xrightarrow{0} \mathbb{Z}) \simeq \mathbb{Z}/\{0\} \simeq \mathbb{Z}$. Similarly $H^0(L_m(\ell_1,\dots,\ell_n)) \simeq \mathbb{Z}$, so we have

$$H^i(L_m(\ell_1,\dots,\ell_n);\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2n-1 \\ \mathbb{Z}_m & \text{if } i \text{ even, } 1 \le i \le 2n-2 \\ 0 & \text{else.} \end{cases}$$

For $G = \mathbb{Q}$, note that $\mathrm{Hom}(\mathbb{Z}, \mathbb{Q}) = \mathbb{Q}$. So the cellular cochain complex is

$$0 \longrightarrow \mathbb{Q} \xrightarrow{0} \mathbb{Q} \xrightarrow{m} \mathbb{Q} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Q} \xrightarrow{m} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \longrightarrow 0.$$

Then for $0 < i < 2n-1$ we have $H^i(L_m(\ell_1, \dots, \ell_n)) = \ker(\mathbb{Q} \xrightarrow{m} \mathbb{Q})/\mathrm{Im}(\{0\}) = \{0\}/\{0\} = 0$ and similarly $\ker(\mathbb{Q} \xrightarrow{0} \mathbb{Q})/\mathrm{Im}(\mathbb{Q} \xrightarrow{m} \mathbb{Q}) \simeq \mathbb{Q}/m\mathbb{Q} \simeq \mathbb{Q}/\mathbb{Q} = 0$. Since $H^0(L_m(\ell_1, \dots, \ell_n)) = H^{2n-1}(L_m(\ell_1, \dots, \ell_n)) = \mathbb{Q}$, we have with these coefficients that

$$H^i(L_m(\ell_1, \dots, \ell_n); \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = 0, 2n-1 \\ 0 & \text{else.} \end{cases}$$

Now for $G = \mathbb{Z}_m$, note that $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_m) = \mathbb{Z}_m$. Thus, the cochain complex is

$$0 \longrightarrow \mathbb{Z}_m \xrightarrow{0} \mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_m \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_m \xrightarrow{0} \mathbb{Z}_m \longrightarrow 0.$$

Then for i even and $2 \le i \le 2n-2$, we have $H^i(L_m(\ell_1, \dots, \ell_n)) = \ker(\mathbb{Z}_m \to 0)/\mathrm{Im}(\mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_m) = \mathbb{Z}_m/\{0\} = \mathbb{Z}_m$. For i odd and $1 \le i < 2n-1$, observe that $H^i(L_m(\ell_1, \dots, \ell_n)) = \ker(\mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_m)/\{0\} = \mathbb{Z}_m/\{0\} \simeq \mathbb{Z}_m$. Similarly, $H^0(L_m(\ell_1, \dots, \ell_n)) = H^{2n-1}(L_m(\ell_1, \dots, \ell_n)) = \mathbb{Z}_m$. Therefore, we have calculated

$$H^i(L_m(\ell_1, \dots, \ell_n); \mathbb{Z}_m) = \begin{cases} \mathbb{Z}_m & \text{if } 0 \le i \le 2n-1 \\ 0 & \text{else.} \end{cases}$$

Finally, let $G = \mathbb{Z}_p$ for p prime. The cochain complex is

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{0} \mathbb{Z}_p \xrightarrow{m} \mathbb{Z}_p \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z}_p \xrightarrow{m} \mathbb{Z}_p \xrightarrow{0} \mathbb{Z}_p \longrightarrow 0.$$

For $\mathbb{Z}_p \xrightarrow{m} \mathbb{Z}_p$, we know this is an isomorphism if $\gcd(m, p) = 1$ or equivalently (since p is prime) if $p \nmid m$. However, if $p \mid m$ clearly this homomorphism is equivalent to the zero map. For the former case, we get for even i and $2 \le i \le 2n-2$ that $H^i(L_m(\ell_1, \dots, \ell_n)) = \ker(\mathbb{Z}_p \to 0)/\mathrm{Im}(\mathbb{Z}_p \xrightarrow{m} \mathbb{Z}_p) = \mathbb{Z}_p/\mathbb{Z}_p = 0$. When i is odd and $1 \le i < 2n-1$, $H^i(L_m(\ell_1, \dots, \ell_n)) = \ker(\mathbb{Z}_p \xrightarrow{m} \mathbb{Z}_p)/\{0\} = \{0\}/\{0\} = 0$. Also, $H^0(L_m(\ell_1, \dots, \ell_n)) = H^{2n-1}(L_m(\ell_1, \dots, \ell_n)) = \mathbb{Z}_p$, which completes the calculations for when $p \nmid m$. When $p \mid m$, $H^i(L_m(\ell_1, \dots, \ell_n)) = \ker(\mathbb{Z}_p \to 0)/\mathrm{Im}(\mathbb{Z}_p \xrightarrow{m} \mathbb{Z}_p) = \mathbb{Z}_p/\{0\} \simeq \mathbb{Z}_p$ or $H^i(L_m(\ell_1, \dots, \ell_n)) = \ker(\mathbb{Z}_p \xrightarrow{m} \mathbb{Z}_p)/\{0\} = \mathbb{Z}_p/\{0\} \simeq \mathbb{Z}_p$; in both cases, $H^i(L_m(\ell_1, \dots, \ell_n)) \simeq \mathbb{Z}_p$. From here, we have

$$H^i(L_m(\ell_1, \dots, \ell_n); \mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & \text{if } i = 0, 2n-1 \\ 0 & \text{else.} \end{cases} \quad \text{if } p \nmid m$$

$$H^i(L_m(\ell_1, \dots, \ell_n); \mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & \text{if } 0 \le i \le 2n-1 \\ 0 & \text{else.} \end{cases} \quad \text{if } p \mid m.$$

This completes the desired calculations. We now confirm that these answers agree with those given by the Universal Coefficient Theorem. We know that $H_i(L_m(\ell_1, \dots, \ell_n) = \mathbb{Z}$ for $i = 0, 2n-1$, $\mathbb{Z}_m$ for i odd and $1 \le i < 2n-1$, and 0 otherwise. Letting $C = L_m(\ell_1, \dots, \ell_n)$, the UCT provides the short exact sequence

$$0 \to \mathrm{Ext}(H_{i-1}(C), G) \to H^i(C; G) \to \mathrm{Hom}(H_i(C), G) \to 0.$$

For any group G, note that $\mathrm{Ext}(\mathbb{Z}, G) = 0$ and $\mathrm{Ext}(G, \mathbb{Q}) = 0$. Furthermore,

- $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}) = 0$,
- $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}) \simeq \mathbb{Z}_m$,
- $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_m) \simeq \mathbb{Z}_m$,
- for $d = \gcd(m, n)$, $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_n) = 0$ if $d = 1, \mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_d$ if $d > 1$,
- $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Q}) = 0$.

For $G = \mathbb{Z}$, we have $H^0(C, \mathbb{Z}) = \mathrm{Hom}(H_0(C), \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$. Also

$$0 \to \mathrm{Ext}(H_0(C), \mathbb{Z}) \to H^1(C; \mathbb{Z}) \to \mathrm{Hom}(H_1(C), \mathbb{Z}) \to 0.$$

Therefore

$$0 \to 0 \to H^1(C; \mathbb{Z}) \to 0 \to 0$$

and so $H^1(C; \mathbb{Z}) = 0$. For i even and $2 \le i \le 2n-2$ observe that we have

$$0 \to \mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}) \to H^i(C; \mathbb{Z}) \to \mathrm{Hom}(0, \mathbb{Z}) \to 0.$$

Therefore
$$0 \to \mathbb{Z}_m \to H^i(C;\mathbb{Z}) \to 0 \to 0$$
and so $H^i(C;\mathbb{Z}) = \mathbb{Z}_m$ for even i in the range $2 \leq i \leq 2n-2$. For i odd and $1 \leq i < 2n-1$, we have
$$0 \to 0 \to H^i(C;\mathbb{Z}) \to 0 \to 0$$
and so $H^i(C;\mathbb{Z}) = 0$ for all i odd such that $i \neq 2n-1$. For $i = 2n-1$, a similar calculation yields $H^i(C;\mathbb{Z}) = \mathbb{Z}$. Hence, by the UCT
$$H^i(L_m(\ell_1,\dots,\ell_n);\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2n-1 \\ \mathbb{Z}_m & \text{if } i \text{ even}, 1 \leq i \leq 2n-2 \\ 0 & \text{else} \end{cases}$$
which is consistent with the previous calculation. For $G = \mathbb{Q}$, we similarly have $H^0(C;\mathbb{Q}) = H^{2n-1}(C;\mathbb{Q}) = \mathbb{Q}$. For $0 < i < 2n-1$, the short exact sequence is
$$0 \to \operatorname{Ext}(\mathbb{Z}_m,\mathbb{Q}) \to H^i(C;\mathbb{Q}) \to 0 \to 0$$
meaning (since $\operatorname{Ext}(\mathbb{Z}_m,\mathbb{Q}) = 0$) that $H^i(C;\mathbb{Q}) = 0$ for $0 < i < 2n-1$. Thus,
$$H^i(L_m(\ell_1,\dots,\ell_n);\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = 0, 2n-1 \\ 0 & \text{else.} \end{cases}$$
as before. Now let $G = \mathbb{Z}_m$. For i even and $2 \leq i \leq 2n-2$, we have
$$0 \to \operatorname{Ext}(\mathbb{Z}_m,\mathbb{Z}_m) \to H^i(C;\mathbb{Z}_m) \to \operatorname{Hom}(0,\mathbb{Z}_m) \to 0.$$
Since $\operatorname{Ext}(\mathbb{Z}_m,\mathbb{Z}_m) \simeq \mathbb{Z}_m$, we have $H^i(C;\mathbb{Z}_m) \simeq \mathbb{Z}_m$ for i even and $2 \leq i \leq 2n-2$. For $i = 1$, we have
$$0 \to \operatorname{Ext}(\mathbb{Z},\mathbb{Z}_m) \to H^1(C;\mathbb{Z}_m) \to \operatorname{Hom}(\mathbb{Z},\mathbb{Z}_m) \to 0$$
meaning
$$0 \to 0 \to H^1(C;\mathbb{Z}_m) \to \mathbb{Z}_m \to 0$$
is exact. Hence, $H^1(C;\mathbb{Z}_m) \simeq \mathbb{Z}_m$ as well. For i odd and $3 \leq i < 2n-1$, the UCT yields
$$0 \to 0 \to H^1(C;\mathbb{Z}_m) \to \mathbb{Z}_m \to 0$$
and so $H^i(C;\mathbb{Z}_m) \simeq \mathbb{Z}_m$ as well. For the final cases, we have that $H^0(C;\mathbb{Z}_m) = H^{2n-1}(C;\mathbb{Z}_m) \simeq \mathbb{Z}_m$. Therefore, the Universal Coefficient Theorem gives
$$H^i(L_m(\ell_1,\dots,\ell_n);\mathbb{Z}_m) = \begin{cases} \mathbb{Z}_m & \text{if } 0 \leq i \leq 2n-1 \\ 0 & \text{else,} \end{cases}$$
which matches the calculation from the cochain complex. Finally, we use the UCT to compute $H^i(C;\mathbb{Z}_p)$. For i even and $2 \leq i \leq 2n-2$, we have
$$0 \to \operatorname{Ext}(\mathbb{Z}_m,\mathbb{Z}_p) \to H^i(C;\mathbb{Z}_p) \to \operatorname{Hom}(0,\mathbb{Z}_p) \to 0.$$
So $H^i(C;\mathbb{Z}_p) \simeq \operatorname{Ext}(\mathbb{Z}_m,\mathbb{Z}_p)$ for all i even, $2 \leq i \leq 2n-2$. If $p \nmid m$, then p being prime means $\gcd(p,m) = 1$ and so $\operatorname{Ext}(\mathbb{Z}_m,\mathbb{Z}_p) = 0$. If $p \mid m$, then $\operatorname{Ext}(\mathbb{Z}_m,\mathbb{Z}_p) \simeq \mathbb{Z}_p$. Now for $i = 1$, observe that
$$0 \to \operatorname{Ext}(\mathbb{Z},\mathbb{Z}_p) \to H^1(C;\mathbb{Z}_p) \to \operatorname{Hom}(\mathbb{Z}_m,\mathbb{Z}_p).$$
and so (since $\operatorname{Ext}(\mathbb{Z},\mathbb{Z}_p) = 0$) $H^1(C;\mathbb{Z}_p) \simeq \operatorname{Hom}(\mathbb{Z}_m,\mathbb{Z}_p)$. If $p \nmid m$, then $H^1(C;\mathbb{Z}_p) \simeq \operatorname{Hom}(\mathbb{Z}_m,\mathbb{Z}_p) = 0$. If $p \mid m$, then $H^1(C;\mathbb{Z}_p) \simeq \operatorname{Hom}(\mathbb{Z}_m,\mathbb{Z}_p) = \mathbb{Z}_p$. Now for $3 \leq i < 2n-1$ and i odd, the short exact sequence is
$$0 \to 0 \to H^i(C;\mathbb{Z}_p) \to \operatorname{Hom}(\mathbb{Z}_p,\mathbb{Z}_m)$$
so again $H^i(C;\mathbb{Z}_p) = 0$ if $p \nmid m$ and $H^i(C;\mathbb{Z}_p) \simeq \mathbb{Z}_p$ otherwise. Also, $H^0(C;\mathbb{Z}_p) \simeq \mathbb{Z}_p$ (by connectedness, or by examining the short exact sequence). Finally,
$$0 \to \operatorname{Ext}(H_{2n-2}(C);\mathbb{Z}_p) \to H^{2n-1}(C;\mathbb{Z}_p) \to \operatorname{Hom}(H_{2n-1},\mathbb{Z}_p) \to 0,$$
which is
$$0 \to 0 \to H^{2n-1}(C;\mathbb{Z}_p) \to \mathbb{Z}_p \to 0.$$

Thus, $H^{2n-1}(C;\mathbb{Z}_p) \simeq \mathbb{Z}_p$ regardless of whether p divides m. Putting these calculations together,

$$H^i(L_m(\ell_1,\dots,\ell_n);\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & \text{if } i = 0, 2n-1 \\ 0 & \text{else.} \end{cases} \quad \text{if } p \nmid m$$

$$H^i(L_m(\ell_1,\dots,\ell_n);\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & \text{if } 0 \le i \le 2n-1 \\ 0 & \text{else.} \end{cases} \quad \text{if } p \mid m.$$

Therefore, the Universal Coefficient Theorem verifies all of the previous calculations of the cohomology groups of $L_m(\ell_1,\dots,\ell_n)$ with coefficients in $G = \mathbb{Z}, \mathbb{Q}, \mathbb{Z}_m, \mathbb{Z}_p$ for p prime. □